

LECTURE 1: OVERVIEW AND ANTECEDENTS

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1. TOPOLOGICAL PRE-HISTORY

The problem of studying manifolds up to homeomorphism/diffeomorphism is an extremely difficult problem. One way to state the problem is as follows. Let $n \geq 0$ be fixed, and consider Mfld_n the topological category of n -manifolds. The objects of this category are smooth, closed¹ n -manifolds and the maps are continuous maps between them. There is a variant of this category where one can consider *smooth* (in other words \mathcal{C}^∞) morphisms instead, we denote this latter category by $\text{Mfld}_n^{\text{sm}}$.

Question 1.0.1. For any given n , describe $\pi_0(\text{Mfld}_n)$ and $\pi_0(\text{Mfld}_n^{\text{sm}})$. In other words, classify n -manifolds up to homeomorphism/diffeomorphism.

Example 1.0.2. Suppose that $n = 2$. Recall that two smooth surfaces are diffeomorphic if and only if they are homeomorphic (this fact remains true for 3-folds). Hence, the inclusion $\text{Mfld}_2^{\text{sm}} \subset \text{Mfld}_2$ induces an isomorphism $\pi_0(\text{Mfld}_2) \cong \pi_0(\text{Mfld}_2^{\text{sm}})$. Furthermore two invariants completely determine the homeomorphism type of a surface: the Euler characteristic (which is an integer ≤ 2) and whether or not it is orientable.

For $n \geq 3$. This is an extremely intractable problem and people who study this kind of thing surely know more than me. For example Milnor first found **exotic spheres**: n -manifolds which are homeomorphic but *not* diffeomorphic to the n -sphere, starting with $n = 7$.

One of the things we do as mathematicians is to recast the problem, in this case of classifying manifolds, in a different way.

Definition 1.0.3. A **cobordism** between two smooth, closed n -manifolds X, Y is a pair (W, f) where W is an $n + 1$ -manifold with boundary and a diffeomorphism:

$$f : X \sqcup Y \rightarrow \partial W.$$

If such a (W, f) exists, we say that X and Y are **cobordant**.

Remark 1.0.4. The relation of cobordism is manifestly coarser than the relation of being diffeomorphic or homeomorphic. There are other variants of this definition; here's a relevant case for us. Suppose that M is a manifold, and $\mathcal{E} \rightarrow M$ is a real vector bundle of rank $2k$. Then an **almost complex structure** is a reduction of the structure group of \mathcal{E} from $\text{GL}(2k, \mathbf{R})$ to $\text{GL}(k, \mathbf{C})$; a **stable almost complex structure** is a complex structure on $\mathcal{E} \oplus \mathcal{O}^n$ for $n \gg 0$. An **(stably) almost complex manifold** is a manifold with an (stable) almost complex structure on its tangent bundle. From this one can define a notion of cobordism in the almost complex setting where the manifold witnessing the cobordism also has an almost complex structure and the diffeomorphism to the boundary is required to respect all the attendant structures.

This maneuver turns out to be ingenious (in many sense of the word), Thom was able to classify manifolds up to cobordism. Indeed, let us write

$$\text{MO}_n = \{X \in \pi_0 \text{Mfld}_n\} / \{X \text{ is equivalent to } Y \text{ iff they are cobordant}\} \sqcup \{\emptyset\}$$

Define

$$\text{MO}_* := \bigoplus_{n \geq 0} \text{MO}_n.$$

¹Compact and without boundary.

This gadget has the structure of a ring:

- (1) the coproduct of manifolds endows MO_* with the structure of an abelian group and the empty manifold acts as the additive identity.
- (2) The cartesian product of manifolds endows MO_* with the multiplicative structure such that:

$$\text{MO}_n \times \text{MO}_m \rightarrow \text{MO}_{n+m};$$

the point: $*$ $\in \text{MO}_0$ acts as the multiplicative identity.

- (3) Furthermore MO_* is an \mathbb{F}_2 -algebra: the manifold $M \times I$ defines a cobordism between $M \sqcup M$ and the empty manifold so that

$$[M] = -[M],$$

in MO_* .

Theorem 1.0.5 (Thom). *The \mathbb{F}_2 -algebra MO_* is free polynomial on generators u_i of dimension i for $i > 1$ and $i \neq 2^r - 1$*

Example 1.0.6. We in fact know some explicit generators for MO_* . When $i = 2j$ we take

$$u_{2j} = [\mathbb{R}P^{2j}].$$

When i is an odd number not of the form $2^r - 1$ we can write it as

$$i = 2^p(2q + 1) - 1 = 2^{p+1}q + 2^p - 1.$$

We define for $m < n$ the quadric hypersurface

$$\text{H}_{n,m} = \{[x_0 : \cdots : x_n], [y_0 : \cdots : y_m] : x_0y_0 + x_1y_1 + \cdots + x_my_m = 0\} \subset \mathbb{R}P^n \times \mathbb{R}P^m;$$

and we take

$$u_i = [\text{H}_{2^{p+1}q, 2^p}].$$

These $\text{H}_{n,m}$'s are also known as **Milnor hypersurfaces**, and we will encounter them again in the context of algebraic geometry.

Theorem 1.0.5 or, rather, its proof was one of the first instance of stable homotopy theory entering the world of manifolds. It gave us one of the most important of classes of (the topologist's notion of) spectra which are called these days called Thom spectra. We could run an entire class on the algebro-geometric notion of (the topologist's notion of) spectra and, in fact, this will come back in the second half of this class. What I would like to do, however, is more geometric. It is clear that the notion of cobordism is helpful as it defines an equivalence relation on the set of manifolds which renders the equivalence classes of them quite computable. It is natural to ask:

Question 1.0.7. What is the algebro-geometric analog of cobordism? Can we classify algebraic varieties (over a ground field k) up to this notion of cobordism?

To address Question 1.0.7, it will be useful to adopt a “parametrized” viewpoint. Suppose that $(\text{Mfld}^{\text{prop}, \partial})^\nabla$ is the (discrete) category of compact manifolds (of any dimension), but possibly with boundary. The prop indicates that we demand the morphisms to be continuous, proper morphisms.

Given a fixed n , we have the “parametrized” version of the category $\text{Mfld}_n^{\text{sm}}(X)$:

- (1) the objects of $\text{Mfld}_n^{\text{sm}}(X)$ are continuous, proper² maps

$$f : Y \rightarrow X,$$

such where Y is of dimension n .

²Recall that proper morphism in point-set topology just means that the inverse image of a compact subset is compact

(2) A morphism is a smooth, continuous map

$$Y \rightarrow Y'$$

over X .

For each X , there is a natural way to endow the set of maps between Y and Y' with a topology such that when $X = *$, then we recover the topological category $\mathbf{Mfld}_n^{\text{sm}}$ which had been defined earlier. Whatever the definition of $'$ is, one gets a functor

$$\mathbf{Mfld}_n : (\mathbf{Mfld}^{\text{prop}, \partial})^\heartsuit \rightarrow \mathbf{Spc} \quad X \mapsto \iota \mathbf{Mfld}_n^{\text{sm}}(X),^3$$

where the functoriality is given by “proper pushforward”. There is a less obvious functoriality of \mathbf{Mfld}_n ; for a manifold X we redefine

$$\mathbf{Mfld}^n(X) := \mathbf{Mfld}_{\dim(X)-n}(X).$$

We can thus consider the vertices of the ∞ -groupoid $\mathbf{Mfld}^n(X)$ as maps $f : Y \rightarrow X$ where Y is a manifold which is of **relative dimension** $-n$. Given a map

$$g : W \rightarrow X,$$

we would like to construct the pullback functor

$$g^* : \mathbf{Mfld}^n(X) \rightarrow \mathbf{Mfld}^n(W) \quad g^*(Y \rightarrow X) = Y \times_X W \rightarrow W.$$

In general, this is not reasonable for two reasons:

- (1) unless g is a transverse to f , the inverse image is not necessarily a manifold;
- (2) the relative dimension of $Y \times_X W \rightarrow W$ is not necessarily $-n$.

If we were able to solve the two problems above, then we can reformulate the cobordism relation in a rather elegant way: the data of a cobordism between two manifolds X and Y of dimension n is “equivalent” to an \mathbf{R} -point of the functor $\mathbf{Mfld}^{-n}(X)$. Indeed, such a thing classifies a continuous, proper map

$$f : Y \rightarrow \mathbf{R},$$

where $\dim Y = n+1$. Now, if 0 and 1 were regular values of f (which is the generic situation, via Sard’s theorem for example), which is equivalent to saying that f is transverse to $\{0\} \sqcup \{1\} \subset \mathbf{R}$ then we will conclude that $f^{-1}(0)$ and $f^{-1}(1)$ are closed manifolds and therefore cobordant. In this situation, we can then define $\text{MO}_n = \text{MO}^{-n}$ by the coequalizer formula

$$\pi_0(\mathbf{Mfld}_n(\mathbf{R})) \rightrightarrows \pi_0(\mathbf{Mfld}_n(*)) \rightarrow \text{MO}_n.$$

In fact, we can define the parametrized version of the above construction:

$$\pi_0(\mathbf{Mfld}^{-n}(X \times \mathbf{R})) \rightrightarrows \pi_0(\mathbf{Mfld}^{-n}(X)) \rightarrow \text{MO}^{-n}(X),$$

which encourages us to think of the the functor $X \mapsto \text{MO}_*(X)$ as a cohomology theory. In fact, the spoiler is that we do have a cohomology theory which is represented by Thom’s bordism spectrum MO .

Remark 1.0.8. The point of view above deviates from the usual blurb in topology about extraordinary cohomology theories in that the primary functoriality is of *pushforward* rather than *pullback*. The formalism of **Borel-Moore homology theories** captures invariants with such functoriality. We will discuss this notion in algebraic geometry.

Remark 1.0.9. In order to build pullbacks, one needs to instead consider the topologically-enriched variant $\mathbf{Mfld}^{\text{prop}, \partial}$ of $(\mathbf{Mfld}^{\text{prop}, \partial})^\heartsuit$. This lets us speak of “replacing a map by a transverse one.”

³The notation ι means that we take the **maximal subgroupoid** of an ∞ /topological-category — discard all the non-invertible morphisms.

2. TOWARDS ALGEBRAIC COBORDISM

The goal of this class is to make the above picture precise, and in the context of algebraic geometry. This is precisely the theory of **algebraic cobordism** as constructed by Levine and Morel.

Theorem 2.0.1 (Levine-Morel). *Let k be a field of characteristic zero. There is a universal oriented Borel-Moore homology theory*

$$\Omega_* : \text{Sch}_k^{\text{prop}} \rightarrow \text{Ab}_*,$$

such that:

- (1) For each $X \in \text{Sch}_k$ $\Omega_n(X)$ has a geometric description in terms of morphisms $f : Y \rightarrow X$ where $Y \in \text{Sm}_k$ of dimension n up to “algebraic cobordism.”
- (2) Let \mathbf{L} be the Lazard ring classifying the universal formal group law, then there is a canonical graded isomorphism

$$\mathbf{L}^* \cong \Omega_*(k).$$

- (3) If $X \in \text{Sch}_k$ and $i : Z \hookrightarrow X$ is a closed immersion with open complement $j : U \hookrightarrow X$ then we have an exact sequence

$$\Omega_*(Z) \xrightarrow{i_*} \Omega_*(X) \xrightarrow{j^*} \Omega_*(U) \rightarrow 0.$$

I hope to cover this theorem in this class and more. In fact, we will construct a version of Ω_* which uses *derived algebraic geometry*. In particular, we will extend, following Lowrey and Schrög, the functor Ω_* to derived k -schemes. The advantage of the derived viewpoint is, of course, to resolve the pesky transversality issues that we have already encountered in topology.

Theorem 2.0.1 has the following amazing consequence on the study of algebraic cycles.

Remark 2.0.2. Let X be a finite type scheme over k and by $Z^r(X)$ we mean the free abelian group generated by cycles

$$i : Z \subset X,$$

where i is a closed immersion, Z is an integral scheme so that Z is a subvariety of X . There is an equivalence relation on $Z^r(X)$ called **rational equivalence**, and the resulting abelian group is called the **Chow group** of X , $\text{CH}^r(X)$. In particular, $\text{CH}^1(X) \cong \text{Pic}(X)$ in great generality (locally factorial). If we redefine

$$\text{CH}_*(X) := \text{CH}^{\dim(X)-*}(X);$$

then we can consider CH_* as a Borel-Moore functor where the primary functoriality is push-forward of cycles along proper maps. In fact, as we will see later, this defines a Borel-Moore homology theory.

Therefore, Theorem 2.0.1 furnishes a transformation

$$\Omega_* \rightarrow \text{CH}_*;$$

which is, roughly, take a cobordism class $[Y \rightarrow X]$ and take the scheme image.

Theorem 2.0.3. *Let k be a field of characteristic zero, then for any smooth k -scheme X the map*

$$\Omega^*(X) \rightarrow \text{CH}^*(X),$$

factors through

$$\Omega^*(X) \otimes_{\mathbf{L}} \mathbf{Z} \rightarrow \text{CH}^*(X),$$

which is an functorial isomorphism in X .

Remark 2.0.4. In topology, the counterpart for Ω_* is the cohomology theory of **complex cobordism** MU_* which admits an analogous map

$$\text{MU}_*(X) \rightarrow H_*(X; \mathbf{Z}).$$

Theorem 2.0.3 is *not* true topology: the map

$$\text{MU}_*(X) \otimes_{\mathbf{L}^*} \mathbf{Z} \rightarrow H_*(X; \mathbf{Z}),$$

is *not* an isomorphism which has to do with the failure of **Landweber's exactness criterion**.

Therefore, if X is smooth and proper then, after identifications using Poincaré duality, get a commutative diagram:

$$\begin{array}{ccc} \Omega^*(X) & \longrightarrow & \text{MU}^{2*}(X(\mathbf{C})) \\ \downarrow & & \downarrow \\ \Omega^*(X) \otimes_{\mathbf{L}^*} \mathbf{Z} & \longrightarrow & \text{MU}^{2*}(X(\mathbf{C})) \otimes_{\mathbf{L}^*} \mathbf{Z} \\ \downarrow \simeq & & \downarrow \\ \text{CH}^*(X) & \longrightarrow & H^*(X, \mathbf{Z}). \end{array}$$

The resulting composite

$$\text{CH}^*(X) \rightarrow \text{MU}^{2*}(X(\mathbf{C})) \otimes_{\mathbf{L}^*} \mathbf{Z} \rightarrow H^*(X, \mathbf{Z}),$$

is known as **Totaro's factorization** of the cycle class map and provides an obstruction to the integral Hodge conjecture. We will discuss this as an application of our formalism.

3. NAIVE COBORDISM

To end off, let us discuss the naive notion of cobordism in algebraic geometry. Suppose that X is a finite type scheme over k . We first define $\mathcal{M}_*^+(X)$ to be the group completion of the following graded monoid (where $[p : Y \rightarrow X]$ indicates isomorphism classes):

$$\mathcal{M}_*(X) := (\{[p : Y \rightarrow X] : Y \text{ is smooth over } k, p \text{ is projective}\}, \sqcup),$$

where the grading is by dimension of Y as a smooth k -scheme. The class of the empty scheme $[\emptyset \rightarrow X]$ serves as the identity. We want to define the so-called **naive cobordism** relation on $\mathcal{M}_*^+(X)$. To begin with suppose that we have a projective morphism

$$q : Y \rightarrow \mathbf{P}^1 \times X.$$

We can form the following diagram where the squares in the top half are all cartesian

$$\begin{array}{ccccc} Y_0 & \longrightarrow & Y & \longleftarrow & Y_\infty \\ \downarrow & & \downarrow q & & \downarrow \\ X & \xrightarrow{0} & \mathbf{P}^1 \times X & \xleftarrow{\infty} & X \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & X & \longleftarrow & \infty \end{array}$$

We say that two X -schemes $V, W \in \text{Sm}_k$ are **naively cobordant** if there are isomorphisms $V \cong Y_0, W \cong Y_\infty$ over X fitting into the above picture. We define

$$\mathcal{R}_*(X) \subset \mathcal{M}_*^+(X),$$

to be the subgroup generated by

$$[V \rightarrow X] - [W \rightarrow X],$$

where V, W are smooth k -schemes which are naively cobordant and we define

$$\Omega_*^{\text{naive}}(X) := \mathcal{M}_*^+(X) / \mathcal{R}_*(X).$$

While this is very nice, it will turn out to not be sufficient for many purposes.

4. EXERCISES

Exercise 4.0.1. *Formulate correctly the notion of a stably almost complex cobordism of stably almost complex manifolds.*

Exercise 4.0.2. *Prove that the cobordism relation is an equivalence relation. Do it for the almost complex case as well.*

Exercise 4.0.3. *Use the classification of (real) closed surfaces to compute MO_2 .*

Exercise 4.0.4. *Recall the **adjunction formula**: if X is a smooth k -variety, and $D \subset X$ is a smooth divisor then we have a linear equivalence of divisors*

$$(K_X + D)|_D = K_D,$$

where K denotes the canonical divisor. Use this and Serre duality to prove: if $p : Y \rightarrow \mathbf{P}^1$ is a proper surjective morphism, then the geometric fibers of p all have the same arithmetic genus.

Exercise 4.0.5. *Prove that a morphism $f : X \rightarrow Y$ is an open immersion if and only if it is étale and radicial.*

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