LECTURE 2: A PRIMER TO DERIVED ALGEBRAIC GEOMETRY I

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The goal of this lecture is to explain how the naive cobordism relation is "not enough." We will also start defining what a derived scheme is and introduce the notion of a flat morphism.

1. Naive cobordism is not enough

The relation of naive cobordism is not sufficient for a good theory of algebraic cobordism. Recall that a **Dedekind scheme** is an integral noetherian scheme such that every local ring is either a field or a discrete valuation ring. Here's a very nice result about them:

Proposition 1.0.1. *Let* X *be a Dedekind scheme and* $f : \mathrm{Y} \to \mathrm{X}$ *be a morphism. Suppose that* Y *is reduced, then* f *is flat if and only if every irreducible component of* Y *dominates* X.

Proof. We may suppose that $\mathrm{X} = \operatorname{Spec} \mathrm{R}$ where R is a Dedekind domain. Recall that a module over a Dedekind domain is flat if and only if it is torsionfree[1] (since being flat is local, we may localize at a maximal idea of A whence it is a discrete valuation ring; one of the characterizations of a valuation ring is the equivalence between a torsionfree module and flat modules). Further supposing that $\mathrm{Y} = \operatorname{Spec} \mathrm{A}$ is an affine scheme, it suffices to prove that as an R-module A is torsionfree.

Let K be the field o fractions of R. Now, the torsion element of A can be expressed as

$$\mathrm{A}_{\mathrm{tors}} = \ker(\mathrm{A} \to \mathrm{A} \otimes \mathrm{K} \qquad a \mapsto a \otimes 1).$$

Furthermore, the closure of the generic fiber $f^{-1}(\operatorname{Spec} \mathrm{K}) \subset \mathrm{Y}$ is isomorphic to $\operatorname{Spec} \mathrm{A}/\mathrm{A}_{\mathrm{tors}}$. We thus conclude that A is torsionfree if and only if the closure of the generic fiber is all of $\mathrm{Y} = \operatorname{Spec} \mathrm{A}$.

We now conclude: if every irreducible component of X dominates Y then we have that $f^{-1}(\eta)$ contains all the generic points of X; the topological closure of $f^{-1}(\eta)$ must then the underlying topological space of X. But X is reduced and so this X as a scheme. Therefore A is flat. Conversely, assume that $\mathrm{Y} \to \mathrm{X}$ is flat so that A is torsionfree. If there exists some irreducible component not dominating X then there exists a component X_α not in the closure of the generic fiber which means that A is not a flat R-algebra. □

Proposition 1.0.2. *Let* X *be a Dedekind scheme and* Y *is reduced. Then any proper, surjective morphism* $f : \mathrm{Y} \to \mathrm{X}$ *be a proper, surjective morphism is flat.*

The next lemma is known as "Euler characteristic is constant in fibers."

Lemma 1.0.3. *Let* $f : \mathrm{Y} \to \mathrm{X}$ *be a projective morphism,* $\mathcal{F} \in \mathbf{Coh}(\mathrm{Y})$ *which is flat over* X *(e.g. if* $\mathrm{Y} \to \mathrm{X}$ *was flat and* $\mathcal{F} = \mathcal{O}_{\mathrm{Y}}$*). Assume that* X *is noetherian. Then the function*

$$\mathrm{X} \to \mathbf{Z} \qquad x \mapsto \chi(\mathrm{Y}_x; \mathcal{F})(:= \Sigma(-1)^i h^i(\mathrm{Y}_x; \mathcal{F})).$$

is a locally constant function.

Proof sketch. Since f is projective, we have a closed immersion $\mathrm{Y} \hookrightarrow \mathbf{P}^n_{\mathrm{X}}$. Hence we may assume that $\mathrm{Y} = \mathbf{P}^n_{\mathrm{X}}$. By Serre vanishing, the Hilbert polynomial agrees with the Euler characteristic and so we may appeal to the result for Hilbert polynomials. □

[1]Recall that an R-module M is torsionfree if and only if the only torsion element, i.e. an element x such that some nonzero $f \in \mathrm{R}$ kills it, is zero.

Remark 1.0.4. Suppose that X_0, X_1 be smooth k-schemes of dimension 1. Then, Lemma 1.0.3 tells us that if X_0 is cobordant to X_1 then they have the same genus. In particular this means that $\Omega_1^{\mathrm{naive}}(k)$ is an abelian group with rank $\geqslant 1$. On the other hand $\mathbf{L}^1 = \mathbf{Z}$ compatibly so with the topological situation. In particular, the topological picture suggests that the smooth genus g curve should be cobordant to $(1-g)[\mathbf{P}^1]$.

Remark 1.0.5. Thinking through the explanation for the failure of naive cobordism to be a reasonable theory, one attempts to find various "fixes." Perhaps one should not demand that the "witnessing cobodism" be a smooth scheme — Proposition 1.0.1 suggests that we should not allow Y to be reduced. This suggests that we should enlarge what mean by a "cobordism cycle" to include nilpotents. Stretching one's imagination, one could perhaps appeal to derived algebraic geometry, which is what we do next.

2. Derived schemes

The basic building blocks of derived schemes are **affine derived schemes**. Just as in conventional algebraic geometry where we define the category of affine schemes as the opposite of the category of commutative rings, it suffices to define what derived rings are. A model for these are **simplicial commutative rings**. The basic idea in defining simplicial commutative rings is to follow the next recipe:

(∗) Consider a commutative ring A. We can write A as a filtered colimit
$$\operatorname*{colim}_{\alpha} \mathrm{A}_\alpha \cong \mathrm{A},$$
where $\mathrm{A}_\alpha \subset \mathrm{A}$ is a finitely presented subring. Supposing that A is finitely presented, we can find as surjection of rings
$$\varphi : \mathbf{Z}[\mathrm{T}_1, \cdots, \mathrm{T}_n] \to \mathrm{A},$$
and a map
$$\mathbf{Z}[\mathrm{T}_1, \cdots, \mathrm{T}_m] \to \mathbf{Z}[\mathrm{T}_1, \cdots, \mathrm{T}_n] \qquad \mathrm{T}_i \mapsto f_i,$$
where the f_i's generate the ideal $\mathrm{I} = \ker(\varphi)$. The ring A is then described as the coequalizer
$$\mathbf{Z}[\mathrm{T}_1, \cdots, \mathrm{T}_m] \rightrightarrows \mathbf{Z}[\mathrm{T}_1, \cdots, \mathrm{T}_n]\mathrm{A} \qquad \mathrm{T}_i \mapsto (f_i, 0).$$

The recipe (∗) tells us that the category of commutative rings $\mathrm{CAlg}^{\heartsuit}$ can be built from the full subcategory $\mathrm{Poly} \subset \mathrm{CAlg}^{\heartsuit}$ spanned by the finitely generated polynomial algebras, i.e., those of the form $\mathbf{Z}[\vec{\mathrm{T}}]$ using only *coequalizers* and *filtered colimits*. In the derived world, filtered colimits remain the same, while coequalizers should be replaced by something more sophisticated. Recall that a simplicial object in an ∞-category $\mathcal{C}$ is a functor
$$\mathrm{X}_\bullet : \Delta^{\mathrm{op}} \to \mathcal{C}.$$
If the colimit exists we denote it by $|\mathrm{X}_\bullet|$ and we call it the **geometric realization**. Diagrams which are filtered and Δ^{op}-shaped are instances of something called **sifted diagrams**, the exact definition of which we do not need[2]. What we need to know is that every sifted diagram can be produced from filtered and Δ^{op}-shaped ones. The higher analog of recipe (∗) should then say:

(∗∗) The ∞-category of derived rings is the free ∞-category generated by Poly under sifted colimits.

Quillen taught us a recipe to describe this ∞-category.

Definition 2.0.1. A **derived ring/simplicial commutative ring/derived affine scheme** is a functor
$$\mathcal{F} : (\mathrm{Poly})^{\mathrm{op}} \to \mathrm{Spc},$$
which converts finite coproducts in Poly to products. This ∞-category is denoted by
$$\mathrm{CAlg} := \mathrm{PSh}_\Sigma(\mathrm{Poly}).$$

[2]A small simplicial set is said to be **sited** if the diagonal map $\mathrm{K} \to \mathrm{K} \times \mathrm{K}$ is cofinal.

We present several ways to think about derived affine schemes.

Remark 2.0.2. For the categorically minded, we have the following result.

Theorem 2.0.3. *[Universal property of* CAlg*] Suppose that* $\mathcal{C}$ *is an* ∞*-category which admits sifted colimits, then the yoneda functor*

$$h : \mathrm{Poly} \to \mathrm{CAlg},$$

induces an equivalence of ∞*-categories:*

$$\mathrm{Fun}_{\mathrm{sift}}(\mathrm{CAlg}, \mathrm{C}) \to \mathrm{Fun}(\mathrm{Poly}, \mathrm{C}),$$

where the left hand side are functors from $\mathrm{CAlg} \to \mathrm{C}$ *which preserves sifted colimits.*

In this way, we make the intuition $(**)$ precise. Basically, the idea is that sifted colimits commutes past finite products; imposing the preservation of finite products turns out to be equivalent to adding sifted colimits.

Remark 2.0.4. Here's a model dependent way to construct the category CAlg which justifies the name "simplicial commutative rings." Begin with the 1-category of commutative rings and consider simplicial objects in commutative rings:

$$\mathrm{SCR} := \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{CAlg}^{\heartsuit}).$$

An object is displayed as a $\{\mathrm{A}_\bullet\}$ where each A_n is a commutative ring. A morphism of commutative rings $\mathrm{A}_\bullet \to \mathrm{B}_\bullet$ is a **weak equivalence** if the underlying map of simplicial sets is a weak equivalence of simplicial sets, i.e., it induces an isomorphism on homotopy groups. The ∞-category CAlg is obtained by formally inverting these weak equivalences; denoting the class of weak equivalences by W we write this equivalence as

$$\mathrm{SCR}[\mathrm{W}^{-1}] \simeq \mathrm{CAlg}. \tag{2.0.5}$$

Remark 2.0.6. Another way to understand a simplicial commutative ring is to unpack its definition as a functor

$$\mathrm{R} : \mathrm{Poly}^{\mathrm{op}} \to \mathrm{Spc}.$$

First there is the **underlying space** of a simplicial commutative ring

$$\mathrm{R}_{\mathrm{spc}} := \mathrm{R}(\mathbf{Z}[\mathrm{T}]).$$

There are canonical maps in Poly given by

$$m' : \mathbf{Z}[\mathrm{T}] \to \mathbf{Z}[\mathrm{T}_1, \mathrm{T}_2] \qquad \mathrm{T} \mapsto \mathrm{T}_1\mathrm{T}_2;$$

$$a' : \mathbf{Z}[\mathrm{T}] \to \mathbf{Z}[\mathrm{T}_1, \mathrm{T}_2] \qquad \mathrm{T} \mapsto \mathrm{T}_1 + \mathrm{T}_2.$$

They induce diagrams

$$m, a : \mathrm{R}_{\mathrm{spc}}^{\times 2} \xleftarrow{\simeq} \mathrm{R}(\mathbf{Z}[\mathrm{T}_1, \mathrm{T}_2]) \xrightarrow{(m')^*, (a')^*} \mathrm{R}_{\mathrm{spc}}.$$

The ***i*-th homotopy group** of a simplicial commutative ring is defined to be $\pi_i(\mathrm{R}_{\mathrm{spc}})$. Using this "parametrized ring structure" on $\mathrm{R}_{\mathrm{spc}}$, the graded abelian group

$$\bigoplus_{i \geqslant 0} \pi_i(\mathrm{R}_{\mathrm{spc}}),$$

acquires the structure of a graded commutative ring.

As usual, it will be useful to define relative notions:

Definition 2.0.7. Let R be a derived ring. The ∞-category of **derived R-algebras** is defined as

$$\mathrm{CAlg}_{\mathrm{R}} := \mathrm{CAlg}_{\mathrm{R}/}.$$

In other words, a derived R-algebra is a derived ring A equipped with an morphism $\mathrm{R} \to \mathrm{A}$.

2.1. **Discrete rings.** The ∞-category Spc contains the discrete category Sets as the full subcategory of those spaces which are 0-truncated, i.e., $\pi_i(\mathrm{X}) = 0$ whenever $i > 0$. The inclusion

$$\mathrm{Sets} \hookrightarrow \mathrm{Spc},$$

admits a left adjoint which sends a space X to $\pi_0(\mathrm{X})$, the set of its connected components. Similarly, we define a derived ring R to be **discrete** or **classical** if the functor $\mathrm{R} : \mathrm{Poly} \to \mathrm{Spc}$ lands in Sets. By the preceding discussion, this means that R uniquely determines a classical ring. Hence we have an adjunction

$$\pi_0 : \mathrm{CAlg} \rightleftarrows \mathrm{CAlg}^{\heartsuit} : \iota. \tag{2.1.1}$$

The canonical unit map

$$\mathrm{R} \to \iota(\pi_0(\mathrm{R})),$$

should be thought of as analogous to the map $\mathrm{R} \to \mathrm{R}_{\mathrm{red}}(\cong \mathrm{R}/\mathrm{I}_{\mathrm{nilp}})$ which "kills nilpotents."

2.2. **Modules over derived rings; Tor-amplitude.** Suppose that R is a simplicial commutative ring. Under the equivalence of (2.0.5), each R determines (up to homotopy) a simplicial object in commutative rings, which we denote by $\mathrm{R}_\bullet$. We can regard this process as picking a "strict model" for R. In this language it is easy to develop a theory of modules. An $\mathrm{R}_\bullet$**-module** is a simplicial abelian group $\mathrm{M}_\bullet$, equipped with simplicial maps

$$\mathrm{R}_\bullet \times \mathrm{M}_\bullet \to \mathrm{M}_\bullet;$$

subject to the usual axioms to be a module; we denote this 1-category by $\mathrm{dgMod}_{\mathrm{R}}$. The ∞-category of R**-modules** is then obtained from $\mathrm{dgMod}_{\mathrm{R}}$ by inverting weak equivalences

$$\mathbf{Mod}_{\mathrm{R}} := \mathrm{dgMod}_{\mathrm{R}}[\mathrm{W}^{-1}].$$

There is a model independent way to define R-modules.

Example 2.2.1. If R is a discrete ring, then $\mathbf{Mod}_{\mathrm{R}}$ is equivalent an ∞-categorical enhancement of the triangulated category of the (unbounded) derived category of R.

Roughly speaking, we consider $\mathrm{R}_{\mathrm{spc}}$ as an $\mathbf{E}_\infty$-ring spectrum, whence we may appeal to the notion of an $\mathrm{R}_{\mathrm{spc}}$-module spectrum. The topologists in the room are free to flesh out this definition. Analogous to the adjunction (2.1.1) we have an adjunction

$$\mathbf{Mod}_{\mathrm{R}} \rightleftarrows \mathbf{Mod}^{\heartsuit}_{\pi_0(\mathrm{R})} \qquad \mathrm{M} \mapsto \pi_0(\mathrm{M}) \cong \pi_0(\mathrm{R}) \otimes^{\mathbf{L}}_{\mathrm{R}} \mathrm{M},$$

where the right adjoint is fully faithful and the left adjoint is displayed; any object in its essential image is called a **discrete** module.

Definition 2.2.2. Let $\mathrm{M} \in \mathbf{Mod}_{\mathrm{R}}$. We say that M has **Tor amplitude** $\leqslant n$ (where n could be ∞) if it for any discrete module N,

$$\pi_i(\mathrm{M} \otimes_{\mathrm{R}} \mathrm{N}) = 0 \qquad \mathrm{i} > \mathrm{n}.$$

We say that M is **flat** if it has Tor-amplitude $\leqslant 0$. We say that an R-algebra S has **Tor amplitude** $\leqslant n$ (resp. flat) if its underlying R-module is.

For the most part, we will be interested in R-modules which are **connective**: $\pi_i(\mathrm{M}) = 0$ for $i < 0$. The next result also gives an equivalent notion of flatness; it follows from Lazard's theorem which (classically) states that flat modules are exactly those which can be written as filtered colimit of finitely generated free modules.

Theorem 2.2.3 (Lazard-Lurie)**.** *Suppose that* $\mathrm{R} \in \mathrm{CAlg}$*. Then the following are equivalent:*

(1) M *can be written as a filtered colimit of finitely generated free modules.*
(2) M *is connective and satisfies (i)* $\pi_0(\mathrm{M})$ *is a flat* $\pi_0(\mathrm{R})$*-module and (ii) the canonical map* $\pi_0(\mathrm{R})$*-module map*

$$\pi_0(\mathrm{M}) \otimes_{\pi_0(\mathrm{R})} \pi_n(\mathrm{R}) \to \pi_n(\mathrm{M}),$$

is an isomorphism.

(3) M *is a connective, flat* R*-module.*

Proof sketch. (1)⇒(2): one easily checks this for M = R and use the fact that the functors in sight all preserve filtered colimits.

(2)⇒(3). Suppose that M satisfies the condition then for any R-module N we have vanishing

$$\mathrm{Tor}_p^{\pi_* \mathrm{R}}(\pi_*(\mathrm{N}), \pi_*(\mathrm{M})) = 0$$

for all $p > 0$ for any R-module N. Therefore, the Tor spectral sequence degenerates to give an isomorphism

$$\mathrm{Tor}_0^{\pi_0 \mathrm{R}}(\pi_q(\mathrm{N}), \pi_0(\mathrm{M})) \xrightarrow{\cong} \pi_q(\mathrm{N} \otimes^{\mathbf{L}} \mathrm{M}).$$

In particular if N is discrete so that $\pi_{\geqslant 1}(\mathrm{N}) = 0$, we have that $\pi_{\geqslant 1}(\mathrm{N} \otimes^{\mathbf{L}} \mathrm{M}) = 0$, hence M is flat.

(3)⇒(2) First we claim that $\pi_0(\mathrm{M})$ is flat. Indeed, for any discrete R-module N (which canonically identifies as a $\pi_0(\mathrm{R})$-module) we have that $\pi_1(\mathrm{N} \otimes^{\mathbf{L}} \mathrm{M}) \cong \mathrm{Tor}_1^{\pi_0(\mathrm{R})}(\mathrm{N}, \pi_0(\mathrm{M}))$ which is zero by assumption. Now, the claim about (ii) follows by an induction using the Tor spectral sequence (which degenerates using the hypotheses).

Now, we want to prove (2)⇒(1). Consider the category $\mathrm{Free}_{/\mathrm{M}}$, i.e., the category of free R-modules mapping to M. It suffices to prove that this category is filtered (which we skip, but requires the classical Lazard theorem) and that the canonical map from the colimit M' to M is an equivalence; to do this we compute its homotopy groups. Since M' is a filtered colimit of free modules we have that $\pi_n(\mathrm{M}') \cong \pi_n(\mathrm{R}) \otimes_{\pi_0(\mathrm{R})} \pi_0(\mathrm{M}')$ since we have alrea proved the (1)⇒(2) direction. Since the analogous isomorphism holds for $\pi_n(\mathrm{M})$ it suffices to prove that $\pi_0(\mathrm{M}') \to \pi_0(\mathrm{M})$ is an isomorphism. This isomorphism then boils down to the classical Lazard's theorem.

□

We could take (3) as the definition of flat modules, but we will need the notion of Tor amplitude later on.

3. Exercises

Exercise 3.0.1. *The inclusion of 1-groupoids into 1-categories* Gpd ↪ Cat *has a left and a right adjoint. Describe them and prove that they are indeed adjoints. The same reasoning works in the context of* ∞*-categories — describe these adjunctions.*

Exercise 3.0.2. *Let* R *be a simplicial commutative ring. Prove that the homotopy ring* $\pi_*(\mathrm{R})$ *has the structure of a graded commutative ring, i.e., the multiplication satisfies*

$$x \cdot y = (-1)^{|x||y|} y \cdot x.$$

Exercise 3.0.3. *Use the result from the previous homework to give an alternative reasoning that the relation* $(1-g)[\mathbf{P}^1] \equiv [\Sigma_g]$ *cannot hold in naive cobordism.*

Exercise 3.0.4. *Let* $\mathrm{R} \in \mathrm{CAlg}^{\heartsuit}$ *and consider the map* $\mathrm{R}[\epsilon]/(\epsilon^2) \to \mathrm{R}$*. Compute the Tor amplitude of* R *as an* $\mathrm{R}[\epsilon]/(\epsilon^2)$*-module.*

Exercise 3.0.5. *Suppose that* $\mathrm{R} \in \mathrm{CAlg}^{\heartsuit}$ *and suppose that* M *is a flat* R*-module. Prove that*

$$\mathrm{Sym}^n_{\mathrm{R}}(\mathrm{M}[1])[n] \simeq \bigwedge_{\mathrm{R}}^{n}(\mathrm{M}) \qquad \bigwedge_{\mathrm{R}}^{n}(\mathrm{M}[1]) \simeq \Gamma^n_{\mathrm{A}}[n],$$

where $\wedge^n$ *indicates the* n*-th exterior algebra, and* Γ^n *indicates the* n*-th divided power algebra.*

Exercise 3.0.6. *Prove that if* R *is a simplicial commutative ring, then for any* $x \in \pi_1(\mathrm{R})$*, we have that* $x^2 = 0$.

References

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