LECTURE 3: A PRIMER TO DERIVED ALGEBRAIC GEOMETRY II

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This is the second introductory lecture on derived algebraic geometry. Here, we will learn how to globalize the preceding construction to obtain a working theory of derived schemes. In our approach, we first obtain the category of derived stacks (though they are not necessarily equipped with an Atlas, i.e., an *Artin* stack) and isolate the category of derived schemes.

1. Open immersions

We have previously discussed the theory of R-modules. As in the classical setting the formation of R-modules enjoys some functoriality properties. Suppose that $\mathrm{R} \to \mathrm{S}$ is a morphim in CAlg which we think of as a morphism

$$f : \operatorname{Spec} \mathrm{S} \to \operatorname{Spec} \mathrm{R},$$

we have an adjunction

$$f^*(\simeq - \otimes_{\mathrm{R}} \mathrm{S}) : \mathrm{Mod}_{\mathrm{R}} \rightleftarrows \mathrm{Mod}_{\mathrm{S}} : f_*(\simeq (-)_{\mathrm{R}}).$$

Already in the classical setting, this can be used to characterize open immersions. As a case-study in derived algebraic geometry, we will explain this.

Definition 1.0.1. Let $\mathrm{R}' \in \mathrm{CAlg}_{\mathrm{R}}$. Then the morphism $f : \mathrm{R} \to \mathrm{R}'$ is said to be **locally finitely presented** if R' is a compact object in $\mathrm{CAlg}_{\mathrm{R}}$. In other words, for any filtered diagram of $\{\mathrm{A}_\alpha\}$ of R-algebras the canonical map

$$\operatorname{colim} \mathrm{Maps}_{\mathrm{R}}(\mathrm{R}', \mathrm{A}_\alpha) \to \mathrm{Maps}_{\mathrm{R}}(\mathrm{R}', \operatorname{colim} \mathrm{A}_\alpha),$$

is an equivalence (of spaces).

This agrees with the classical definition of being locally finitely presented. To motivate the definition of an open immersion and the cotangent complex. Here are some facts that one has to know about the cotangent complex for classical rings:

Lemma 1.0.2. *Suppose that $f : \mathrm{R} \to \mathrm{S}$ is a morphism in $\mathrm{CAlg}^{\heartsuit}$ and locally finitely presented. Then f is étale if and only if $\mathbf{L}_{\mathrm{R}/\mathrm{S}}$ is étale.*

The invention of the cotangent complex was one of the starting points of derived algebraic geometry; indeed the above reformulates a property about a morphism of schemes purely in *homological term.* In fact, we also have

Lemma 1.0.3. *Suppose that $f : \mathrm{R} \to \mathrm{S}$ is a morphism in $\mathrm{CAlg}^{\heartsuit}$ and locally finitely presented. Then f is smooth if and only if $\mathbf{L}_{\mathrm{R}/\mathrm{S}}$ has Tor-amplitude $[0, 0]$, i.e., it is discrete and is a projective module in degree 0. In this case, $\mathbf{L}_{\mathrm{R}/\mathrm{S}} \simeq \Omega_{\mathrm{R}/\mathrm{S}}[0]$.*

From this point of view, the next lemma of Grothendieck's is quite natural: it produces the notion of an open immersion from knowledge of étale morphisms

Lemma 1.0.4. *Let $f : \mathrm{X} \to \mathrm{Y}$ be a morphism of classical schemes, locally of finite presentation. Then the following are equivalent:*

(1) *f is an open immersion,*
(2) *f is an étale monomorphism, i.e., an étale morphism such that the diagonal map $\mathrm{X} \to \mathrm{Y} \times_{\mathrm{X}} \mathrm{Y}$ is an isomorphism.*

Lemma 1.0.5. *Suppose that* $\mathrm{R} \to \mathrm{S}$ *is a morphism in* $\mathrm{CAlg}^{\heartsuit}$ *which is locally finitely presented. Then the following are equivalent*

(1) *the induced morphism* $f : \operatorname{Spec} \mathrm{S} \to \operatorname{Spec} \mathrm{R}$ *is an open immersion.*
(2) *the functor* $f_* : \mathbf{Mod}_{\mathrm{S}} \to \mathbf{Mod}_{\mathrm{R}}$ *is fully faithful.*

Proof. We leave the direction (1) ⇒ (2) to the reader (look at the case $\mathrm{R} \to \mathrm{R}[\frac{1}{f}]$. To prove the converse, we consider the (invertible) transformation $f^*f_* \to \mathrm{id}$ on two objects S and the cotangent complex $\mathbf{L}_{\mathrm{S}/\mathrm{R}}$:

$$\mathrm{S} \otimes_{\mathrm{R}} \mathrm{S} \xrightarrow{\simeq} \mathrm{S} \qquad \mathrm{S} \otimes_{\mathrm{R}} \mathbf{L}_{\mathrm{S}/\mathrm{R}} \to \simeq \mathbf{L}_{\mathrm{S}/\mathrm{R}}.$$

Since the first map is an isomorphism, the diagonal is an isomorphism, hence it is a monomorphism on Spec. On the other hand, since the cotangent complex is stable under *derived* base change we have equivalences:

$$\mathbf{L}_{\mathrm{S}/\mathrm{R}} \otimes_{\mathrm{R}} \mathrm{S} \simeq \mathbf{L}_{\mathrm{S} \otimes_{\mathrm{R}} \mathrm{S}/\mathrm{S}} \simeq \mathbf{L}_{\mathrm{S}/\mathrm{S}} \simeq 0,$$

whence $\mathrm{R} \to \mathrm{S}$ is étale □

Definition 1.0.6. A morphism of derived rings $\mathrm{R} \to \mathrm{S}$ is said to be an **open immersion** if it is locally finitely presented and

$$f_* : \mathbf{Mod}_{\mathrm{S}} \to \mathbf{Mod}_{\mathrm{R}}$$

is fully faithful.

2. A quick recap of the cotangent complex

Let R be a fixed discrete base commutative ring. Recall that we have a functor

$$\mathrm{CAlg}_{\mathrm{R}}^{\heartsuit} \to \mathrm{Mod}_{\mathrm{R}}^{\heartsuit} \qquad \mathrm{S} \mapsto \Omega^1_{\mathrm{S}/\mathrm{R}}.$$

Let me remark that this is not such a good viewpoint — $\Omega^1_{\mathrm{S}/\mathrm{R}}$ is naturally an S-module and we have forgotten the S-module structure and only viewed it as an R-module. A more robust point of view is to consider it as a section of a certain fibration. Nonetheless, we push on, restricting the functor Ω^1 to $\mathrm{Poly}_{\mathrm{R}}$ we have the functor which, concretely:

$$\mathbf{Z}[\mathrm{T}_1, \cdots, \mathrm{T}_n] \mapsto \mathbf{Z}[\mathrm{T}_1, \cdots, \mathrm{T}_n]\{d\mathrm{T}_1, \cdots d\mathrm{T}_n\}.$$

Employing the universal property from Lecture 2 we obtain a functor which preserves sifted colimits:

$$\mathbf{L}_{-/\mathrm{R}} : \mathrm{CAlg}_{\mathrm{R}} \to \mathrm{Mod}_{\mathrm{R}},$$

which is concretely computed as taking a free resolution of S as a free R-algebra and taking geometric realization. Here are computations which one can perform:

Lemma 2.0.1. *Let* R *be a discrete commutative ring.*

(1) *there is natural augmentation map*

$$\mathbf{L}_{\mathrm{S}/\mathrm{R}} \to \Omega^1_{\mathrm{S}/\mathrm{R}}$$

which induces an isomorphism

$$\pi_0(\mathbf{L}_{\mathrm{S}/\mathrm{R}}) \cong \Omega^1_{\mathrm{S}/\mathrm{R}}.$$

(2) *If* $\mathrm{S} = \mathrm{R}/\mathrm{I}$ *where* I *is an ideal of* R, *then* $\pi_0(\mathbf{L}_{\mathrm{S}/\mathrm{R}}) = 0$ *and* $\pi_1(\mathbf{L}_{\mathrm{S}/\mathrm{R}}) \cong \mathrm{I}/\mathrm{I}^2$.

The next proposition sums up all we need about the cotangent complex, which one can construct as a functor on derived rings

$$\mathbf{L}_{-/\mathrm{R}} : \mathrm{CAlg}_{\mathrm{R}} \to \mathrm{Mod}_{\mathrm{R}}$$

Proposition 2.0.2. *Let* R *be a derived ring and* $\mathrm{R} \to \mathrm{S}$ *an* R*-algebra. Then the following hold:*

(Derived base change) *Let* $\mathrm{R} \to \mathrm{R}'$ *be a morphism, then there is a canonical equivalence*

$$\mathbf{L}_{\mathrm{S}/\mathrm{R}} \otimes^{\mathbf{L}}_{\mathrm{R}} \mathrm{R}' \xrightarrow{\simeq} \mathbf{L}_{\mathrm{R}' \otimes^{\mathbf{L}}_{\mathrm{R}} \mathrm{S}/\mathrm{R}}$$

(Künneth) *Suppose that* S' *is another* R*-algebra then there is a canonical equivalence*

$$\mathbf{L}_{S\otimes^{\mathbf{L}}_{R}S'/R} \simeq \mathbf{L}_{S/R} \otimes^{\mathbf{L}}_{R} S' \oplus \mathbf{L}_{S'/R} \otimes^{\mathbf{L}}_{R} S.$$

(Transitivity) *Suppose that* $S \to S'$ *is a morphism, then we have a sequence*

$$\mathbf{L}_{S/R} \otimes^{\mathbf{L}}_{S} S' \to \mathbf{L}_{S'/R} \to \mathbf{L}_{S'/S}"$$

3. Globalization, derived stacks and derived schemes

One way to think of a (classical) scheme is in terms of its functor of points. Indeed, if X is a scheme, we get a functor

$$h_X : \mathrm{CAlg}^{\heartsuit}(= \mathrm{Aff}^{\mathrm{op}}) \to \mathrm{Sets},$$

which subsequently reconstructs the scheme. The functor h_X satisfies some nice properties. For starters, given rings R, R' the canonical map

$$h_X(R \times R') \to h_X(R) \times h_X(R')$$

is an isomorphism of sets (in other words, the Σ-condition). Furthermore, suppose that $\{R \to R_\alpha\}_{\alpha\in I}$ is an **(finitary) fpqc cover**[1], then we have an equalizer diagram

$$h_X(R) \to \prod_{I} h_X(R_\alpha) \rightrightarrows \prod_{I\times I} h_X(R_\alpha \otimes_R R_{\alpha'}).$$

Indeed, one can show that the category of schemes embeds into the category of **fpqc sheaves** on CAlg:

$$\mathrm{Sch} \hookrightarrow \mathrm{Shv}_{\mathrm{fpqc}}(\mathrm{CAlg}).$$

To define derived schemes, our task is to emulate this in the derived setting. We have already defined the notion of flat. We say that a morphism $f : R \to S$ of derived rings is **faithfully flat** if S is a flat as an R-module and the induced map $\mathrm{Spec}\,\pi_0(S) \to \mathrm{Spec}\,\pi_0(R)$ is surjective as topological spaces. From this, there is an obvious notion of an fpqc cover.

Definition 3.0.1. The ∞-category of **derived stacks** dStk is the full subcategory of $\mathrm{PSh}_\Sigma(\mathrm{CAlg})$ of those Σ-presheaves $\mathcal{F}$ satisfying fpqc descent:

- for any fpqc cover $\{R \to R_\alpha\}_{\alpha\in I}$ the canonical map

$$\mathcal{F}(R) \to \lim_{\Delta} \mathcal{F}(\check{C}(\widetilde{R}/R)),$$

  is invertible.

Example 3.0.2. It is often useful to consider **prestacks**, simply an object $\mathcal{F} \in \mathrm{PSh}(\mathrm{CAlg})$. By general nosense the inclusion dStk admits a left adjoint

$$L_{\mathrm{fpqc}} : \mathrm{PSh}(\mathrm{CAlg}) \to \mathrm{dStk},$$

called **stackification**

The statement of faithfully flat descent can be phrased in the following way:

Lemma 3.0.3. *The fpqc topology on* CAlg *is subcanonical.*

Proof. This follows from the usual statement of faithfully flat descent:

- if $A \to B$ is faithfully flat, then the canonical map

$$A \to \lim \check{C}(B/A),$$

  is invertible.

∎

[1]This means:

(1) the indexing set I is finite,
(2) the map $R \to R_\alpha$ is flat (but not necessarily locally finitely presented),
(3) the induced map $R \to \prod R_\alpha (:= \widetilde{R})$ is surjective in Spec.

Hence, the Yoneda embedding factors through dStk

$$\mathrm{CAlg} \to \mathrm{dStk}.$$

We are now ready to globalize

Definition 3.0.4. Suppose that $f : \mathcal{U} \to \mathcal{X}$ is a morphism of derived stacks.

- if f is a morphism of affine derived stacks, then we know what it means for f to be an open immersion.
- Suppose that $\mathcal{X} = \operatorname{Spec} \mathrm{R}$ is affine. We say that f is an **open immersion** if there exists a family
$$\{\mathcal{U}_\alpha = \operatorname{Spec} \mathrm{S}_\alpha \to \mathcal{U}\}_\alpha$$
such that
$$\sqcup \mathcal{U}_\alpha \to \mathcal{U}$$
is an epimorphism[2] of derived stacks and the composite
$$\operatorname{Spec} \mathrm{S}_\alpha \to \mathcal{U},$$
is an open immersion.
- Finally, in the general case, we say that f is an open immersion if for any test point $\mathrm{T}(= \operatorname{Spec} \mathrm{R}) \to \mathcal{X}$, the base change
$$\mathrm{T} \times_{\mathcal{X}} \mathcal{Y} \to \mathrm{T}$$
is an open immersion.

Definition 3.0.5. A derived stack $\mathcal{X}$ is said to be a **derived scheme** if there exists a collection $\{\mathcal{U}_\alpha \to \mathcal{X}\}$ where each $\mathcal{U}_\alpha = \operatorname{Spec} \mathrm{R}_\alpha$ is an affine derived scheme such that the map

$$\sqcup_\alpha \mathcal{U}_\alpha \to \mathcal{X}$$

is an epimorphism. The ∞-category of derived schemes is the full subcategory

$$\mathrm{dSch} \subset \mathrm{dStk},$$

spanned by derived schemes.

Recall that if R is a ring, the map $\mathrm{R} \to \mathrm{R}_{\mathrm{red}}$ is the unit of an adjunction beteween reduced rings and rings. On Spec, this gives a closed immersion

$$\mathrm{X}_{\mathrm{red}} \hookrightarrow \mathrm{X},$$

and the adjunction between reduced schemes and schemes are reversed. Similarly, we have an adjunction

$$\iota : \mathrm{Sch} \rightleftarrows \mathrm{dSch} : (-)_{\mathrm{cl}},$$

compatible with the affine case earlier; note that the functor which extracts the classical locus is a *right adjoint*. The canonical morphism induced by the counit transformation $\mathrm{X}_{\mathrm{cl}} \to \mathrm{X}$ witnesses the **classical locus** or the **underlying classical scheme** of the derived scheme X. We say that a derived scheme is **classical** if the map $\mathrm{X}_{\mathrm{cl}} \to \mathrm{X}$ is invertible. The next lemma is almost immediate.

[2]A epimorphism in the ∞-category of derived stacks is given as follows: they are exactly morphism $f : \mathcal{X} \to \mathcal{Y}$ such that the canonical map

$$\operatorname*{colim}_{\Delta^{\mathrm{op}}} \check{\mathrm{C}}(\mathcal{X}/\mathcal{Y}) \to \mathcal{Y},$$

is invertible.

4. A worked example

To create interesting derived schemes we exploit the following phenomenon:

- the inclusion $\mathrm{Sch} \hookrightarrow \mathrm{dSch}$ *does not preserve pullbacks.*

Indeed, this is visible on fact that the derived tensor product and the tensor product are manifestly very different. Suppose that we have a cospan of derived schemes (possibly classical) $X \to Y \leftarrow Z$, then we get the following situation:

$$(X \times_Y Z)_{cl} \to X \times_Y^{\mathbf{L}} Z$$

where the codomain is the pullback computed in derived schemes. If X, Y, Z are classical then $(X \times_Y Z)_{cl}$ is the pullback computed in classical schemes.

4.1. **Loop space of points.** Suppose that $S = \operatorname{Spec} R$ be a derived affine scheme and consider the projection map $\pi : \mathbf{A}^1_S \to S$ and also the 0-section map

$$0 : S \to \mathbf{A}^1_S.$$

We define $\Omega_0\mathbf{A}^1$ by the (derived) cartesian square

$$\begin{array}{ccc} \Omega_0\mathbf{A}^1 & \longrightarrow & S \\ \downarrow & & \downarrow 0 \\ S & \xrightarrow{0} & \mathbf{A}^1. \end{array}$$

Let us compute (the derived ring of functions on) $\Omega_0\mathbf{A}^1$. First, we note that the inclusion functor $\mathrm{CAlg}_R{}^{op} \hookrightarrow \mathrm{dSch}_R$ preserves pullbacks[3] Hence we are left to compute the pushout in CAlg_R.

(4.1.1)
$$\begin{array}{ccc} R[x] & \longrightarrow & R \\ \downarrow & & \downarrow \\ R & \longrightarrow & \mathcal{O}(\Omega_0\mathbf{A}^1). \end{array}$$

In derived algebraic geometry, the forgetful functor

$$\mathrm{CAlg}_R \to \mathrm{Mod}_R$$

admits a left adjoint

$$\mathrm{LSym} : \mathrm{Mod}_R \to \mathrm{CAlg}_R.$$

whence preseves colimits. We have the pushout diagram in R-modules

$$\begin{array}{ccc} R\{x\} & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & R\{x\}[1], \end{array}$$

which gets taken to the diagram (4.1.1) (after identifying the appropriate maps), whence we conclude that

$$\Omega_0\mathbf{A}^1 \cong \operatorname{Spec} \mathrm{LSym}(R\{x\}[1]).$$

In the class after the next, we will encounter more examples.

Of course the usual pullback is just S again and we have an identification

$$S = \Omega_0\mathbf{A}^1_{cl} \hookrightarrow \Omega_0\mathbf{A}^1.$$

[3]This is analogous to the classical fact of course that the pullback in schemes can be computed by a "tensor product construction." We note that this functor need not preserve pushouts so that pushouts of schemes is not necessarily computed by a pullback of rings.

In the "locally ringed space with an enhanced structure sheaf" formalism, we can think of $\Omega_0\mathbf{A}^1_{\mathrm{cl}}$ as enhancing the structure sheaf of S with the simplicial S-algebra $\mathrm{LSym}(\mathrm{R}\{x\}[1])$. We can make this more explicit: a result of Illusie gives us a canonical identiification:

$$\mathrm{LSym}(\mathrm{R}\{x\}[1]) \cong \bigoplus_{n\geqslant 0} \Sigma^n \bigwedge_{\mathrm{R}}^{n} \mathrm{R}\{x\}.$$

5. Exercises

Exercise 5.0.1. *Prove Lemma 2.0.1.*

Exercise 5.0.2. *Prove Proposition 2.0.2.*

Exercise 5.0.3. *Let* $\mathrm{A} \to \mathrm{B}$ *be a map of* $\mathbf{F}_p$*-algebras and consider the Frobenius map* $\mathrm{Frob} : \mathrm{A} \to \mathrm{A}$ *on* A. *We say that* B *is a* **relatively perfect A-algebra** *if the map*

$$\mathrm{B} \otimes^{\mathbf{L}}_{\mathrm{Frob},\mathrm{A}} \mathrm{A} \to \mathrm{B}$$

is an equivalence. Prove that any such B *satisfies:*

$$\mathbf{L}_{\mathrm{B}/\mathrm{A}} \simeq 0.$$

Exercise 5.0.4. *Suppose that* $\mathrm{A} \to \mathrm{B}$ *is a morphism in* CAlg. *We say that* B *is* **finitely presented** *if it is in the smallest subcategory of* $\mathrm{CAlg}_{\mathrm{A}}$ *which is closed under finite colimits and contains* $\mathrm{A}[x]$. *Prove that any finitely presented* A*-algebra is locally finitely presented. Furthermore, prove that any locally finitely presented* A*-algebra is a retract of a finitely presented* A*-algebra. Show, by a counterexample, that the two notions need not coincide.*

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