

## LECTURE 4: ORIENTED BOREL-MOORE THEORIES

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We take a diversion from derived algebraic geometry to discuss some classical invariants of schemes and introduce the notion of oriented Borel-Moore functors.

### 1. DIGRESSION: CHOW GROUPS AND CHOW HOMOLOGY

The design of algebraic cobordism as a theory is based on the Chow groups as a theory — in the literature this is often called **Chow homology**. Let us go into this digression. Let  $k$  be a field which, for simplicity, we assume that  $k$  is algebraically closed.

**Remark 1.0.1.** Here's what being algebraically closed buys us. Suppose that  $X, Y$  are  $k$ -varieties so that they are, in particular, integral. Then  $X \times_k Y$  are still varieties [Stacks, Tag 05P3]. This is of course not the case if  $k$  admits a non-trivial Galois extension (which is a  $k$ -variety of dimension 0)  $L/k = \text{Spec } L \times_k \text{Spec } L \cong \sqcup_G \text{Spec } L$ . The point here is that the product of integral schemes may fail to be integral if they are not over an algebraically closed field. Being integral is useful for computing dimension: if  $A$  is a  $k$ -algebra which is furthermore an integral domain, then the dimension of  $X = \text{Spec } A$  is simply the transcendental degree of  $\text{Frac}(A)$ ; this follows from Noether normalization. This latter fact, however, does not need  $k$  to be algebraically closed.

Let  $X$  be a finite type  $k$ -scheme of equidimension  $d$ . We let  $0 \leq i \leq d$  and set  $q := d - i$ . We have

$$(1.0.2) \quad Z^i(X) = Z_q(X) := \mathbf{Z}[Z \subset X],$$

where  $Z$  is an **elementary cycle of codimension  $i$ /dimension  $q$** , i.e., a closed subvariety  $Z \hookrightarrow X$ ; being a variety automatically entails that  $Z$  is an integral, separated scheme of finite type over  $k$ . In particular  $Z$  is reduced.

**Remark 1.0.3.** The equation (1.0.2) suggests that algebraic cycles are of a different nature from singular homology cycles — there is an “automatic” version of Poincaré duality which does not require that  $X$  is proper. In this sense, Chow homology is more naturally a Borel-Moore homology, an idea which we will explain later.

**Example 1.0.4.** We have the following sampler of agreement with some familiar objects

- (1)  $Z^1(X) = \text{Div}(X)$ , the group of Weil divisors on  $X$ .
- (2)  $Z_0(X) = Z^d(X)$  is the group of **zero cycles** on  $X$ , these are linear combinations of reduced points of the scheme  $X$ , i.e., a point of  $x \in X_{\text{red}}$ .

The first basic operations that we can perform on  $Z_q(X)$  is **proper pushforward**: suppose that  $f : X \rightarrow Y$  is a proper morphism and let  $Z \subset X$  be an elementary cycle of dimension  $q$ , then  $f(Z) \subset Y$  is a closed subscheme. In case  $\dim(Z) = \dim(f(Z))$ , then  $f|_Z : Z \rightarrow f(Z)$  is a finite dominant morphism, whence its degree is well defined. We define:

$$f_*(Z) = \begin{cases} \deg(Z/f(Z)) [f(Z)] & \text{if } \dim(f(Z)) = q \\ 0 & \text{otherwise.} \end{cases}$$

Extending by linearity we then obtain a map

$$f_* : Z_q(X) \rightarrow Z_q(Y).$$

This shows that it is quite easy to define proper pushforwards.

**1.1. Rational equivalence.** The relation that we want to impose on cycles is called rational equivalence. To set this up, let us recall the following result on the dimension of irreducible components of intersections.

**Lemma 1.1.1.** *Let  $k$  be an algebraically closed field. Suppose that  $X$  is a smooth  $k$ -scheme of dimension  $d$ , and suppose that  $V, W$  are elementary cycles of dimension  $m$  and  $n$ . Write*

$$V \cap W = \cup_j T_j,$$

*where each  $T_j$  is irreducible. Then  $\dim(T_j) \geq m + n - d$ .*

*Proof sketch.* The method used is called **reduction to the diagonal**; this is a classical technique that is fundamental in intersection theory. The observation is based on the following pullback square (for now, underived):

$$\begin{array}{ccc} V \cap W & \longrightarrow & X \\ \downarrow & & \downarrow \Delta \\ V \times W & \longrightarrow & X \times X \end{array}$$

The pullback diagram above is an instance of pullback along a regular immersion (in the language of the next lecture, quasi-smooth) since  $X$  is a smooth  $k$ -scheme.

Hence to compute the dimension of the components of  $V \cap W$  we are reduced to the following situation. Suppose that  $R$  is a domain and  $f \in R$  (so that  $(f)$  is a regular sequence) and suppose that  $\mathfrak{p} \subset R$  is a prime ideal such that  $V(\mathfrak{p})$  is dimension  $d$ , then we claim:

- each irreducible component of  $V(\mathfrak{p}) \cap V(f)$  is dimension  $\geq d - 1$ .

To compute this, it suffices to compute the dimension around  $y = \mathfrak{q} \subset T \subset |V(\mathfrak{p}) \cap V(f)|$  where  $y$  is a closed point in an irreducible component  $T$  such that  $y$  is not contained in any other irreducible component (these are points with the largest local dimension). The local ring at  $y$  is given by  $A_{\mathfrak{q}}/(\mathfrak{p} + f)$ . By Krull's principal ideal theorem we can compute the dimension of this ring as  $\geq d = (\dim(A_{\mathfrak{q}}/(\mathfrak{p}))) - 1$ , but this number is the codimension of  $y \in Y$  which is actually the dimension of  $Y$  whenever  $Y$  is integral and  $y$  is a closed point. This is as desired.  $\square$

**Definition 1.1.2.** Suppose that  $V, W \subset X$  are as in Lemma 1.1.1. Then an **excess component** of  $V \cap W$  is an irreducible component  $T_j \subset V \cap W$  which is of dimension  $> m + n - d$ . We say that  $V, W$  **intersect properly** if there  $V \cap W$  does not have any non-empty excess component, i.e., any component is of dimension  $\leq m + n - d$ .

With this result at hand, consider the following situation:  $X$  is a variety,  $W \subset X \times \mathbf{P}^1$  is a closed subvariety of dimension  $k + 1$  and assume that  $a, b$  are distinct closed points of  $\mathbf{P}^1$ . Then we note that  $X \times a$  and  $X \times b$  intersects  $W$  properly if and only if

$$\dim(W \cap X \times a) \leq k \quad \dim(W \cap X \times b) \leq k.$$

In this situation we say that the cycles  $[W_a], [W_b] \in Z_k(X)$  are **rationally equivalent**. It makes sense to consider the group of cycles of dimension  $k$  modulo rational equivalences; this is the **Chow group of dimension  $k$ -cycles on  $X$** :

$$\mathrm{CH}_k(X) := Z_k(X) / \{\text{cycles rationally equivalent to zero}\}.$$

We define the graded abelian group:

$$\mathrm{CH}_*(X) := \bigoplus_{i \geq 0} \mathrm{CH}_i(X).$$

Furthermore, proper pushforward respects the rational equivalence relation, and thus give a proper morphism  $f : Y \rightarrow X$  we get a morphism

$$f_* : \mathrm{CH}_*(Y) \rightarrow \mathrm{CH}_*(X).$$

Furthermore if  $f : Y \rightarrow X$  is a flat, equidimensional morphism of dimension  $d$ , then we get a pullback morphism:

$$f^! : \mathrm{CH}_*(X) \rightarrow \mathrm{CH}_{*+d}(Y).$$

The pullback and pushforward, when defined, satisfy a base-change formula  $g^! f_* = f_* g^!$  for the obvious cartesian square. Thus far, everything that is said is not something holds up to rational equivalence.

Here's a motivation for this notion. In general, suppose that  $Y, X$  are smooth and quasi-projective, then we want to say that *any* morphism  $f : Y \rightarrow X$  defines a pullback

$$f^! : \mathrm{CH}_*(X) \rightarrow \mathrm{CH}_{*+\dim(Y)-\dim(X)}(Y).$$

Suppose that  $X$  is projective, then there's a nice formula for this; denote by  $\pi_Y : X \times Y \rightarrow Y$  the projection morphism which is projective since  $X$  is and  $\pi_X : X \times Y \rightarrow X$  the other projection which is flat since  $X$  is smooth. Then

$$f^!(Z) = \pi_{Y*}(\Gamma_f \cdot \pi_X^!(Z)).$$

If you are into derived categories, this looks like the “Fourier-transform.” The  $\cdot$  is known as the intersection product and it is take in the  $\mathrm{CH}_*(X \times Y)$ . Everything else is defined so it remains to produce a ring structure on this graded abelian group, via the intersection product.

**1.2. The intersection product.** Suppose again that  $X$  is a smooth  $k$ -scheme and say  $V, W \subset X$  are integral subvarieties of  $X$ . According to Serre, the **intersection product of  $V \cdot W$**  is given by the following recipe:

- (1) write  $V \cap W = \cup T_i$  where each  $T_i$  is irreducible.
- (2) Define

$$e_{T_i} = \sum (-1)^i \mathrm{length}_{\mathcal{O}_{X, T_i}} \mathrm{Tor}_i^{\mathcal{O}_{X, T_i}}(\mathcal{O}_{W, T_i}, \mathcal{O}_{V, T_i}).$$

This number is the **intersection mutliplicity of  $T_i$  in  $V \cap W$** .

- (3) Then the cycle is denoted by

$$V \cdot W = \sum e_{T_i} [T_i]$$

Serre than proves:

**Theorem 1.2.1.** *There is a product:*

$$\cdot : \mathrm{CH}_p(X) \otimes \mathrm{CH}_q(X) \rightarrow \mathrm{CH}_{\dim(X)-p-q}(X),$$

*which agrees with the above formula in the case of proper intersections. The product satisfy associativity, graded commutativity.*

The content of Theorem 1.2.1 is the moving lemma: we can move two cycles so that they intersect properly.

## 2. ORIENTED BOREL-MOORE FUNCTORS

In this section, we give a reformulation of Levine-Morel's notion of an oriented Borel-Moore theory. Chow homology turns out to be an example (left to the reader to fill in details!). For now, we will not keep track of the attendant grading.

**2.1. Categories of spans.** Let  $\mathcal{C}$  be an  $\infty$ -category with finite limits and equipped with two classes of morphisms  $(B, F)$  where  $B$  stands for “backward” and  $F$  stands for “forward.” From this, we can construct the  $\infty$ -category  $\mathbf{Corr}(\mathcal{C}, B, F)$  whose objects are objects in  $\mathcal{C}$  and whose morphisms are spans

$$X \xleftarrow{f} Y \xrightarrow{g} Z,$$

where  $f$  is in  $B$  and  $g$  is in  $F$ . A functor out of this into another  $\infty$ -category  $\mathcal{D}$  (spaces, abelian groups etc.) is then displayed as:

$$\mathbf{Corr}(\mathcal{C}, B, F) \rightarrow \mathcal{D} \quad g_* f^! : E(X) \rightarrow E(Y) \rightarrow E(Z).$$

**Remark 2.1.1.** We state some motivation for notation. The pushforward we take are always for proper or projective morphism so  $f_*$  should be thought of as being the same as  $f_!$ . The pullbacks are only for lci morphisms — these come in two types: a regular immersion or (locally) a vector bundle projection. In either case, they are some kind of “shifted”  $f^*$  which one usually denotes by  $f^!$ .

The composition law in **Corr** encodes the base change formula in an elegant manner. Suppose that we have spans

$$S \leftarrow C \rightarrow T \quad T \leftarrow D \rightarrow U.$$

Then composition is obtained by taking the pullback and forming the diagram:

$$\begin{array}{ccccc} & & A & & \\ & f \swarrow & & \searrow g & \\ C & & & & D \\ h \swarrow & & i \searrow & j \swarrow & \searrow k \\ S & & T & & U \end{array}$$

Classically, one would take  $A$  “up to isomorphism” so that the resulting gadget will be a category. However the technology of  $\infty$ -categories allows us to speak of  $A$  “up to coherent homotopy” so we need not do this.

For this lecture at least,  $\mathcal{C}$  will be a 1-category and the resulting span category will be a  $(2, 1)$ -category. In any case to say that the composite  $E(S) \rightarrow E(T) \rightarrow E(U)$  is homotopic to the induced map  $E(S) \rightarrow E(U)$  is equivalent to asking for an homotopy:

$$j^! i_* \simeq g_* f^!.$$

**2.2. Chern class operators.** Furthermore, we would like a theory of Chern classes. Let us remind ourselves of the topological situation. In more familiar terms, the **first Chern class** of a complex line bundle  $\mathcal{L} \rightarrow X$  is an element

$$c_1(\mathcal{L}) \in H^2(X; \mathbf{Z}).$$

It acts on the homology of  $X$  via the cap product

$$\cap c_1(\mathcal{L}) : H_n(X; \mathbf{Z}) \rightarrow H_{n-2}(X; \mathbf{Z}).$$

In this way, we should regard the first Chern class of a line bundle as a degree  $-2$ -endomorphism of the homology groups, which we call the **Chern class operator**

$$\tilde{c}_1(\mathcal{L}) : H_*(X; \mathbf{Z}) \rightarrow H_{*-2}(X; \mathbf{Z}),$$

Inspired by the theory of Chow homology, we will take a slightly different point-of-view using Borel-Moore homology: for any oriented manifold (possibly open), we we have the Poincaré duality isomorphism

$$H^n(X; \mathbf{Z}) \cong H_{\dim(X)-n}^{\text{BM}}(X, \mathbf{Z}),$$

whence the Chern class operator is dual to cup product:

$$\tilde{c}_1(\mathcal{L}) : H_*^{\text{BM}}(X; \mathbf{Z}) \rightarrow H_{*-2}^{\text{BM}}(X; \mathbf{Z}).$$

Denote by  $\pi_0 \text{Pic}_{\mathbf{C}}$  the Picard group functor, assigning to a topological space  $X$  the group (under  $\otimes$ ) of complex line bundles on  $X$ . We obtain a transformation

$$\pi_0 \text{Pic}_{\mathbf{C}} \Rightarrow \text{End}_{-2}(H_*(-; \mathbf{Z})),$$

obtain from the universal case of  $X = \mathbf{CP}^\infty$ , the moduli space of  $\mathbf{C}$ -line bundles. The Chern class operator further enjoys the following compatibilities for the various functorialities on Borel-Moore homology:

(1) for a proper map  $f : X \rightarrow Y$ , and  $\mathcal{L} \rightarrow Y$  we have

$$\tilde{c}_1(\mathcal{L}) \circ f_* = f_* \circ \tilde{c}_1(f^* \mathcal{L}).$$

(2) for a smooth map (surjective submersion in the manifold context)  $f : X \rightarrow Y$  we have

$$\tilde{c}_1(f^* \mathcal{L}) \circ f^! = f^! \circ \tilde{c}_1(\mathcal{L}).$$

**2.3.  $\mathbf{Corr}_{\text{Pic}}$ .**<sup>1</sup> For safety,  $\text{Sch}_S$  indicates quasi-projective  $S$ -schemes. To encode all the possible functorialities we utilize the following  $(2,1)$ -category, which we denote by  $\mathbf{Corr}_{\text{Pic}}(\text{Sch}_S, \text{smooth}, \text{proj})$  (we write  $\mathbf{Corr}_{\text{Pic}}$  if the context is clear):

(objects) An object is a  $S$ -scheme  $X$ .

(morphisms) A morphism from  $X$  to  $Y$  is a span displayed as

$$\begin{array}{ccc} & Z & \\ (f, \mathcal{L}) \swarrow & & \searrow g \\ X & & Y \end{array}$$

Here,  $f$  is smooth,  $g$  is projective and  $\mathcal{L} \in \text{Pic}(Z)$ .

(composition) To define the composition law: suppose that we want to compose the spans:

$$S \xleftarrow{(h, \mathcal{L})} C \xrightarrow{i} T \quad T \xleftarrow{(j, \mathcal{L}')} D \xrightarrow{k} U$$

We construct the following larger span:

$$\begin{array}{ccccc} & & A & & \\ & (f, i^* \mathcal{L}') \swarrow & & \searrow g & \\ & C & & D & \\ (h, \mathcal{L}) \swarrow & & i \searrow & & \searrow k \\ S & & T & & U \end{array}$$

and the left arrow  $A \rightarrow S$  is labelled by

$$(h \circ f, f^* \mathcal{L} \otimes i^* \mathcal{L}').$$

**Proposition 2.3.1.**  $\mathbf{Corr}_{\text{Pic}}$  forms a  $(2,1)$ -category.

**Definition 2.3.2.** Let  $\mathcal{C}$  be an  $\infty$ -category with products. An  $\mathcal{C}$ -valued oriented Borel-Moore functor is a product-preserving functor

$$A : \mathbf{Corr}_{\text{Pic}}(\text{Sch}_S, \text{smooth}, \text{proj}) \rightarrow \mathcal{C} \quad X \mapsto A(X)$$

Let us unpack some features of a  $\mathcal{C}$ -valued oriented Borel-Moore functor:

(push) if  $f : Y \rightarrow X$  is a proper morphism, we have a pushforward functor  $f_* : A(Y) \rightarrow A(X)$ ; this structure is obtained by restriction along

$$\text{Sch}_S \rightarrow \mathbf{Corr}_{\text{Pic}}(\text{Sch}_S, \text{smooth}, \text{proj}) \quad f : X \rightarrow Y \quad X = X \xrightarrow{f} Y.$$

(pull) if  $f : Y \rightarrow X$  is a smooth morphism, thought of as an “unmarked” arrow  $X \xleftarrow{(f, 0)} Y = Y$  we get a pullback  $f^* : A(X) \rightarrow A(Y)$ ; this structure is obtained by restriction along

$$\text{Sch}_S^{\text{op}} \rightarrow \mathbf{Corr}_{\text{Pic}}(\text{Sch}_S, \text{smooth}, \text{proj}) \quad f : X \rightarrow Y \quad Y \xleftarrow{f} X = X.$$

<sup>1</sup>Some of the ideas here are joint with A. Khan.

(pic) we have a map of monoidal groupoids (equivariance for  $\otimes$  versus  $\circ$ ):

$$c_X : (\text{Pic}(X), \otimes) \rightarrow (\mathbf{Corr}_{\text{Pic}}(X, X), \circ) \quad \mathcal{L} \mapsto X \xleftarrow{=, \mathcal{L}} X \xrightarrow{=} X,$$

as one can easily verify by hand. We denote the map by  $c_X$  to remind ourselves that this is a kind of “Chern class.” Indeed, we get an induced map

$$c_{A(X)} : (\text{Pic}(X), \otimes) \rightarrow \text{Maps}(A(X), A(X)).$$

Furthermore, we have the following compatibilities:

(push-pic) if  $f : X \rightarrow Y$  is a proper morphism thought of as a map  $X = X \xrightarrow{f} Y$  then we have a commutative diagram (which the reader is encouraged to check!).

$$\begin{array}{ccccc} \text{Pic}(Y) & \xrightarrow{c_Y} & \mathbf{Corr}(Y, Y) & \xrightarrow{f^*} & \mathbf{Corr}(X, Y) \\ \downarrow f^* & & & & \downarrow = \\ \text{Pic}(X) & \xrightarrow{c_X} & \mathbf{Corr}(X, X) & \xrightarrow{f_*} & \mathbf{Corr}(X, Y). \end{array}$$

After taking  $A$ , this gives an equivalence:

$$f_* \circ c(f^* \mathcal{L}) \simeq c(\mathcal{L}) \circ f_*$$

(pull-pic) if  $f : X \rightarrow Y$  is a smooth morphism thought of as a map  $Y \xleftarrow{f} X = X$  then we have a commutative diagram (which the reader is encouraged to check!).

$$\begin{array}{ccccc} \text{Pic}(Y) & \xrightarrow{c_Y} & \mathbf{Corr}(Y, Y) & \xrightarrow{f_*} & \mathbf{Corr}(Y, X) \\ \downarrow f^* & & & & \downarrow = \\ \text{Pic}(X) & \xrightarrow{c_X} & \mathbf{Corr}(X, X) & \xrightarrow{f^*} & \mathbf{Corr}(Y, X). \end{array}$$

After taking  $A$ , this gives an equivalence:

$$f^* \circ c(\mathcal{L}) \simeq c(f^* \mathcal{L}) \circ f^*$$

**Remark 2.3.3.** There is also a variant for derived schemes, but we have not explained what smooth and proper morphism are. This is the subject of the next lecture.

### 3. EXERCISES IN THE THEORY OF VIRTUAL DIMENSION

In this sequence of exercises, we develop the theory of virtual (relative) dimension. All schemes in sight are derived. Here’s a notation:  $R \in \text{CAlg}^\heartsuit$  and  $M$  is a discrete  $R$ -module then  $\ell(M) := \sup\{n : M \text{ has a descending chain of length } n\}$  denotes the length of  $M$  as an  $R$ -module. In particular if  $R$  is a derived ring and  $M$  is an  $R$ -module we write  $h_i(R)$  for the length of the  $\pi_0(R)$ -module  $\pi_i(R)$ .

**Definition 3.0.1.** Suppose that  $f : Y \rightarrow X$  is a morphism of derived schemes, We define the **relative virtual dimension** of  $f$  (which is either an integer,  $+\infty$  or  $-\infty$ ) to be

$$\text{virt.dim}(f) := \chi(\mathbf{L}_f) = \sum_{i \geq 0} (-1)^i h_i(\mathbf{L}_f).$$

The **virtual dimension** of a derived scheme  $R$  is the relative virtual dimension of the canonical map  $f : X \rightarrow \text{Spec } \mathbf{Z}$ .

**Exercise 3.0.2.** Prove, from first principles and the properties of the cotangent complex, that the following derived rings have these virtual dimensions:

(1) Let  $X$  be a discrete, smooth  $k$ -scheme and  $f : X \rightarrow \text{Spec } k$  is the structure map. Then

$$\text{virt.dim}(f) = +\dim X.$$

- (2) More generally, if  $f : X \rightarrow Y$  is a smooth morphism of relative dimension  $d$  (taken, as definition, the rank of the locally free sheaf  $\Omega_{X/Y}$ ), then

$$\text{virt.dim}(f) = +d.$$

- (3) Let  $I = (f_1, \dots, f_c)$  be a regular system of parameters in a discrete ring  $R$ , and let  $i : \text{Spec } R/I \hookrightarrow \text{Spec } R$  be the closed immersion, then

$$\text{virt.dim}(i) = -c.$$

- (4) Let  $i_\epsilon : \text{Spec } k \hookrightarrow \text{Spec } k[T]/(T^2)$  be the closed immersion induced by the map  $k[T]/(T^2) \rightarrow k$ . Then

$$\text{virt.dim}(i) = +1.$$

Compute in general the virtual dimension of  $\text{Spec } k \hookrightarrow \text{Spec } k[T]/(T^n)$ .

**Exercise 3.0.3.** [Base change] Suppose that we have a cartesian diagram of derived schemes

$$\begin{array}{ccc} X \times_Z Y & \longrightarrow & X \\ \downarrow f' & & \downarrow f \\ Y & \longrightarrow & Z. \end{array}$$

Prove that  $\text{virt.dim}(f) = \text{virt.dim}(f')$ .

**Exercise 3.0.4.** [Additivity] Suppose that  $X \xrightarrow{f} Y \xrightarrow{g} Z$  is a sequence of morphisms of derived schemes. Prove

$$\text{virt.dim}(g \circ f) = \text{virt.dim}(g) + \text{virt.dim}(f).$$

**Exercise 3.0.5.** [Local nature] Let  $R$  be a ring. Prove that

$$\text{virt.dim}(R) = \sup_{\mathfrak{m} \in \text{MaxSpec}(R)} \text{virt.dim}(R_{\mathfrak{m}}).$$

**Exercise 3.0.6.** [Products] Suppose that  $f : X \rightarrow S, g : Y \rightarrow S$  are derived schemes. Prove that

$$\text{virt.dim}(X \times_S Y) = \text{virt.dim}(f) + \text{virt.dim}(g).$$

**Exercise 3.0.7.** [Virtual Krull] Suppose that  $A$  is a discrete ring and  $f \in A$ . Assume that  $A$  is noetherian, then prove (without appealing to Krull's principal ideal theorem):

$$\text{virt.dim}(A/f) \geq \text{virt.dim}(A) - 1.$$

Conclude that if  $f$  is a nonzero divisor then

$$\text{virt.dim}(A/f) = \text{virt.dim}(A) - 1.$$

Hint: it suffices to prove that the cotangent complex of  $A \rightarrow A/f$  is of tor-amplitude  $[0, 2]$ .

## APPENDIX A. COMPARISON OF BOREL-MOORE FUNCTORS

We will work exclusively with  $\text{Sch}_k$  which stands for the category of quasi-projective  $k$ -schemes. We compare the formulation of Borel-Moore functors presented above versus the one found in [LM07, Definition 2.1.2] where we ignore gradings and denote pullbacks by  $f^!$  and pushforwards by  $f_*$ . For simplicity we let  $\mathcal{C}$  be an abelian category and we also forget about the gradings involved. Oriented Borel Moore functors considered in Definition 2.3.2 will be called Borel-Moore functors and forms a category  $\mathbf{BM}$ , while those considered in *loc. cit.* are called Borel-Moore functors of **LM type** and assembles into a category denoted by  $\mathbf{BM}_{\text{LM}}$ . We note that we have restricted ourselves to a 1-categorical setting. Note that the product-preserving condition in the definition above translates exactly to the additivity condition of [LM07, Definition 2.1.1].

**Construction A.0.1.** Suppose that  $A \in \mathbf{BM}_{\text{LM}}$ , then we want to construct  $T(A) \in \mathbf{BM}$  in the following way. For any  $X \in \text{Sch}_k$ , we set  $T(A)(X) = A(X)$ . Give a span  $X \xleftarrow{(f, \mathcal{L})} Z \xrightarrow{Y}$ , we can express it as a composite of morphisms in  $\mathbf{Corr}_{\text{Pic}}$  as a sequence of spans:

$$X \xleftarrow{(f, \emptyset)} Z \xrightarrow{=} Z \xleftarrow{(\emptyset, \mathcal{L})} Z \xrightarrow{=} Z \xleftarrow{(\emptyset, \emptyset)} Z \xrightarrow{g} Y.$$

We define  $T(A)$  on specific morphisms:

- (1) for  $X \xleftarrow{(f, \emptyset)} Z = Z$ , we set  $f^! : A(X) \rightarrow A(Z)$ ;
- (2) for  $Z \xleftarrow{(\emptyset, \mathcal{L})} Z \xrightarrow{=} Z$  we set  $\tilde{c}_1(\mathcal{L}) : A(Z) \rightarrow A(Z)$ ;
- (3) for  $Z = Z \xrightarrow{g} Y$  we set  $g_* : A(Z) \rightarrow A(Y)$ .

**Lemma A.0.2.** The Construction A.0.1 specifies a Borel-Moore functor. Therefore we have a functor

$$T : \mathbf{BM}_{\text{LM}} \rightarrow \mathbf{BM}.$$

*Proof.* As in Construction A.0.1 we can factor any span in  $\mathbf{Corr}_{\text{Pic}}$  in terms of the three basic types of maps. It suffices to verify that their composites against one another is well defined whenever  $A$  satisfies the axioms listed in [LM07, Definition 2.1.2]:

- (1) Suppose that  $X \xleftarrow{g} Z$  and  $Z \xleftarrow{h} W$  are composable smooth morphisms, which are thought of as spans  $X \xleftarrow{(g, \emptyset)} Z$  and  $Z \xleftarrow{(h, \emptyset)} W$ , then we need to check that  $h^! \circ g^! = (gh)^!$ . This follows from (A1) in [LM07, Definition 2.1.2].
- (2) A similar check can be made for composable projective morphisms and using (D1).
- (3) Suppose that we have two line bundles on  $\mathcal{L}, \mathcal{L}'$  on  $Z$ , then we need to prove that  $\tilde{c}_1(\mathcal{L}) \circ \tilde{c}_1(\mathcal{L}') = \tilde{c}_1(\mathcal{L}' \circ \mathcal{L})$ , which is (A5).
- (4) Suppose that we have a smooth morphism  $g$  thought of as a span  $X \xleftarrow{(g, \emptyset)} Z = Z$  and a proper morphism  $f$  thought of as  $W \xleftarrow{(\emptyset, \emptyset)} W \xrightarrow{f} X$ . Then composition is equivalent to the base change formula of (A2).
- (5) Suppose that  $g$  is a proper morphism thought of as a span  $Z \xleftarrow{(\emptyset, \emptyset)} Z \xrightarrow{g} Y$  and  $\mathcal{L}$  is a line bundle on  $Y$  thought of as a span  $Y \xleftarrow{(\emptyset, \mathcal{L})} Y = Y$ . Then, forming the span

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow^{(\emptyset, g^* \mathcal{L})} & & \searrow_g & \\ & Z & & Y & \\ \swarrow_{(\emptyset, \emptyset)} & & \searrow_g & \swarrow_{(\emptyset, \mathcal{L})} & \searrow_{=} \\ Z & & Y & & Y \end{array},$$

we see that we are demanding:  $\tilde{c}_1(\mathcal{L}) \circ g_* = g_* \circ \tilde{c}_1(g^* \mathcal{L})$  which is (A3).

- (6) Analogously, suppose that  $f$  is a smooth morphism  $Z \rightarrow X$  and  $\mathcal{L}$  is a line bundle on  $X$ , form the span

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow_{(f, \emptyset)} & & \searrow_{=} & \\ & X & & Z & \\ \swarrow_{(\emptyset, \mathcal{L})} & & \searrow_{=} & \swarrow_{(f, \emptyset)} & \searrow_{=} \\ X & & X & & Z \end{array},$$

we see that we are demanding  $f^* \circ \tilde{c}_1(\mathcal{L}) = \tilde{c}_1(f^* \mathcal{L}) \circ f^*$



(7) The other possible composites are trivially satisfied.

□

On the other hand, we have already unpacked the structure of a Borel-Moore functor and seen that it does define naturally Borel-Moore functor of LM type. Evidently, this gives us:

**Theorem A.0.3.** *There is an equivalence of (abelian) categories  $\mathbf{BM} \simeq \mathbf{BM}_{\text{LM}}$ .*

## REFERENCES

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