LECTURE 7: THE DEFINITION OF ALGEBRAIC COBORDISM

ELDEN ELMANTO

In this class, we define (derived) algebraic cobordism following Parker Lowrey and Timo Schrüg, perform some "external construction" and "internal computations" to the theory.

1. Overview for the next few weeks

We will work towards proving the following result of Levine and Morel's.

Theorem 1.0.1. *Let k be a field of characteristic zero, then the canonical map $\mathbf{L}_* \to \Omega_*(\operatorname{Spec} k)$ classifying the formal group law on algebraic cobordism is an isomorphism.*

We will follow the approach of Quillen's which has a chance of working over fields of positive characteristics. En route to this, we will prove also prove the following result, which is due to Levine and Morel and Lowrey and Schrüg in the derived setting.

Theorem 1.0.2. *Let k be a field of characteristic zero, then Ω_* assembles into the universal Borel-Moore homology theory.*

The universality of Ω_* lets us define, among other things, natural transformations

$$\Omega_* \to \mathrm{CH}_*$$

and

$$\Omega_* \to \mathrm{G}_0,$$

here G_0 is the G**-theory group** — which is the Grothendieck group of coherent sheaves on a derived scheme.

2. Quasi-smooth algebraic cobordism

Definition 2.0.1. Let X be a quasi-projective derived k-scheme. For $d \in \mathbf{Z}$, we denote the group of **effective bordism cycles of virtual dimension** d (or just a **cycle** if the context is clear)

$$\mathcal{M}_d(\mathrm{X}) := \{[f : \mathrm{Y} \to \mathrm{X}] : \mathrm{Y} \text{ is an irreducible quasi-smooth } k\text{-scheme of virtual dimension } d,\ f \text{ is proper}\}.$$

We denote by $\mathcal{M}_d^+(\mathrm{X})$ its group completion, it is the group of **bordism cycles of virtual dimension** d. We form the graded ring

$$\mathcal{M}_*(\mathrm{X})^+ := \bigoplus_{d \in \mathbf{Z}} \mathcal{M}_d(\mathrm{X})^+,$$

where $\sqcup$ is addition and $\times^{\mathbf{L}}_{\mathrm{X}}$ is the multiplication.

Now we impose relations. Suppose that Y is a quasi-smooth derived scheme of virtual dimension $d+1$ and suppose that we have a proper morphism $f : \mathrm{Y} \to \mathrm{X} \times \mathbf{P}^1$. This time, consider the derived pullback square:

$$\begin{array}{ccccc}
\mathrm{Y}_0 & \longrightarrow & \mathrm{Y} & \longleftarrow & \mathrm{Y}_\infty \\
\downarrow & & \downarrow q & & \downarrow \\
\mathrm{X} & \xrightarrow{0} & \mathbf{P}^1 \times \mathrm{X} & \xleftarrow{\infty} & \mathrm{X} \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathbf{P}^1 & \longleftarrow & \infty
\end{array}$$

In this situation, Y_0, Y_1 are manifestly quasi-smooth whose virtual dimension are both d. In this case we say that Y_0 and Y_1 are **quasi-smooth cobordant** and the relation is **quasi-smooth cobordism**. Define the subgroup

$$\mathcal{R}_*(X) \subset \mathcal{M}_*(X)^+$$

generated by elements

$$[Y_0 \to X] - [Y_\infty \to X]$$

where Y_0 and Y_∞ are quasi-smooth cobordant. We define the group

$$\mathcal{M}_*(X)^+/\mathcal{R}_*(X) := \Omega_*^{\mathrm{qsm}}(X)$$

to be the **quasi-smooth cobordism group** of X.

2.1. **Pushforward.** Suppose that $f : X \to X'$ is a proper morphism, then we immediately have

$$\mathcal{M}_*^{\mathrm{qsm}}(X) \to \mathcal{M}_*^{\mathrm{qsm}}(X') \qquad [Y \to X] \to [Y \to X \to X']$$

We can postcompose quasi-smooth cobordisms

$$W \to X \times \mathbf{P}^1 \to X' \times \mathbf{P}^1,$$

whence the pushforward descends to a degree-preserving map, the **proper pushforward**:

$$f_* : \Omega_*^{\mathrm{qsm}}(X) \to \Omega_*^{\mathrm{qsm}}(X')$$

2.2. **Pullbacks.** This is where the technology of derived algebraic geometry provides an advance.

Lemma 2.2.1. *Suppose that* X, Y *are smooth* S*-schemes, then any morphism* $f : X \to Y$ *is necessarily quasi-smooth.*

Proof. We have the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \quad \swarrow & \\ & S & \end{array},$$

whence a cofiber sequence

$$f^*\mathbf{L}_{Y/S} \to \mathbf{L}_{X/S} \to \mathbf{L}_f$$

from which we deduce that the $\mathbf{L}_f$ has Tor-amplitude $[0,1]$ since $\mathbf{L}_{X/S}, \mathbf{L}_{Y/S}$ are discrete by assumption, whence we are done by the previous result. □

So suppose that $f : X \to Y$ is a quasi-smooth morphism of derived schemes (in particular this covers the case when f is smooth and also when f is a morphism between smooth schemes) of virtual dimension d and suppose that we have an effective cobordism cycle $[Z \to Y]$ of degree n (which is some integer), we form the following diagram where, for everyone's sake, we have labelled the relevant arrows by virtual dimension

$$\begin{array}{ccc} Z \times_Y X & \longrightarrow & Z \\ \downarrow & & \downarrow \\ X & \xrightarrow{d} & Y \\ & \searrow & \downarrow \\ & & \operatorname{Spec} k \end{array} \quad (Z \to \operatorname{Spec} k \text{ labelled } n)$$

.

Since virtual dimensions are stable under pullbacks, $\mathrm{Z} \times_{\mathrm{Y}} \mathrm{X} \to \mathrm{Z}$ is virtual dimension d, hence the derived scheme $\mathrm{Z} \times_{\mathrm{Y}} \mathrm{X}$ is of virtual dimension $n + d$. Furthermore, $\mathrm{Z} \times_{\mathrm{Y}} \mathrm{X} \to \mathrm{Z}$ is quasi-smooth since being quasi-smooth is stable under base change, whence $\mathrm{Z} \times_{\mathrm{Y}} \mathrm{X} \to \mathrm{Z}$ is indeed quasi-smooth. So we define a map

$$\mathcal{M}_n(\mathrm{Y})^+ \to \mathcal{M}_{n+d}(\mathrm{X})^+ \qquad [\mathrm{Z} \to \mathrm{Y}] \mapsto [\mathrm{Z} \times_{\mathrm{Y}} \mathrm{X} \to \mathrm{X}].$$

The quasi-smooth cobordism relation is preserved by pullbacks:

Lemma 2.2.2. *Suppose that* $\mathrm{Z}_0, \mathrm{Z}_1 \to \mathrm{Y}$ *are quasi-smooth cobordant, then so is* $\mathrm{Z}_0 \times_{\mathrm{Y}} \mathrm{X}$ *and* $\mathrm{Z}_1 \times_{\mathrm{Y}} \mathrm{X}$.

Proof. Indeed, if $\mathrm{W} \to \mathrm{Y} \times \mathbf{P}^1$ is the quasi-smooth cobordism, then $\mathrm{W} \times_{\mathrm{X} \times \mathbf{P}^1} \mathrm{Y} \times \mathbf{P}^1$ is a quasi-smooth k-scheme which defines a quasi-smooth cobordism between $\mathrm{Z}_0 \times_{\mathrm{Y}} \mathrm{X}$ and $\mathrm{Z}_1 \times_{\mathrm{Y}} \mathrm{X}$ by transitivity of pullbacks. □

This lets us define, for any quasi-smooth morphism of derived schemes of virtual dimension d the **quasi-smooth pullback**

$$f^! : \Omega^{\mathrm{qsm}}_*(\mathrm{X}) \to \Omega^{\mathrm{qsm}}_{*+\dim(f)}(\mathrm{X}')$$

Lemma 2.2.3. *Suppose that we have a cartesian diagram of derived schemes where f is proper and g is*

$$\begin{array}{ccc} \mathrm{Y}' & \xrightarrow{g'} & \mathrm{X}' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathrm{Y} & \xrightarrow{g} & \mathrm{X}. \end{array}$$

Then we have a canonical identification

$$g^! f_* = (g')^! f'_*.$$

Proof. Suppose that $[\mathrm{Z} \to \mathrm{X}']$ is a cobordism cycle, then we have that

$$\begin{aligned} (g)^! f_*([\mathrm{Z} \to \mathrm{X}']) &= (g')^*([\mathrm{Z} \to \mathrm{X}]) \\ &= (g')^*([\mathrm{Z} \to \mathrm{X}]) \\ &= [\mathrm{Z} \times_{\mathrm{X}} \mathrm{Y} \to \mathrm{Y}] \\ &= [\mathrm{Z} \times_{\mathrm{X}'} \mathrm{X}' \times_{\mathrm{X}} \mathrm{Y} \to \mathrm{Y}] \\ &= f'_*(g')^! [\mathrm{Z} \to \mathrm{X}]. \end{aligned}$$

□

2.3. **Chern class operators.** Intuitively, the first Chern class of a line bundle measures its complexity in terms of the geometry of the zero's of a generic section. Recall from last class that if $\mathcal{E}$ is a locally free sheaf on a derived scheme X, we can construct the derived scheme $\mathrm{X}(\mathcal{E})$ which is the derived self-intersection of X as the zero section in $\mathcal{V}(\mathcal{E}^\vee)$. We denote the zero section of bundles as $z_{\mathcal{E}} : \mathrm{X} \to \mathcal{V}(\mathcal{E}^\vee)$. In this situation, we note that $z_{\mathcal{E}}$ is quasi-smooth and the cotangent complex of is just $\mathcal{E}$ shifted in degree 1.

Definition 2.3.1. Let $\mathcal{L}$ be a line bundle on X, then **first Chern class operator of** $\mathcal{L}$ is the map

$$\widetilde{c_1}(\mathcal{L}) : \Omega^{\mathrm{qsm}}_*(\mathrm{X}) \to \Omega^{\mathrm{qsm}}_{*-1}(\mathrm{X}) \qquad [\mathrm{Y} \to \mathrm{X}] \mapsto z^!_{\mathcal{L}} z_{\mathcal{L}*}([\mathrm{Y} \to \mathrm{X}])$$

More generally, if $\mathcal{E}$ is a vector bundle on X of rank d, then the **Euler class operator of** $\mathcal{E}$ is the map

$$\widetilde{\chi}(\mathcal{E}) : \Omega^{\mathrm{qsm}}_*(\mathrm{X}) \to \Omega^{\mathrm{qsm}}_{*-d}(\mathrm{X}) \qquad [\mathrm{Y} \to \mathrm{X}] \mapsto z^!_{\mathcal{E}} z_{\mathcal{E}*}([\mathrm{Y} \to \mathrm{X}]).$$

Example 2.3.2. Suppose that X is a quasi-smooth derived scheme. Then we have the cycle $[\mathrm{X} = \mathrm{X}]$. By Lemma 2.2.3 we deduce the equality

$$\widetilde{c_1}(\mathcal{L})([\mathrm{X} = \mathrm{X}]) = [\mathrm{X}(\mathcal{L}) \to \mathrm{X}].$$

Hence, we see that $\widetilde{c_1}(\mathcal{L})([\mathrm{X} = \mathrm{X}])$ literally computes the derived self-intersection.

Lemma 2.3.3. *Supppose that $s : \mathrm{X} \to \mathcal{L}$ is a section. Then we have a natural identification of degree -1-endomorphisms.*

$$\widetilde{c_1}(\mathcal{L}) = s_0^! s_* : \Omega_*^{\mathrm{qsm}}(\mathrm{X}) \to \Omega_{*-1}^{\mathrm{qsm}}(\mathrm{X})$$

Proof. Let $[f : \mathrm{Y} \to \mathrm{X}]$ be a cycle. More generally consider two sections s, s' and following derived cartesian square

$$\begin{array}{ccc} \mathrm{W} & \longrightarrow & \mathrm{Y} \times \mathbf{P}^1 \\ \downarrow & & \downarrow{\scriptstyle \widetilde{s} \circ (f \times \mathrm{id})} \\ \mathrm{X} \times \mathbf{P}^1 & \xrightarrow{z} & \mathbf{V}((\mathcal{O}(1) \otimes \pi^*\mathcal{L})^\vee). \end{array}$$

Here $\widetilde{s}$ is the section defined by $s \otimes x_\infty + s' \otimes x_0$.

By the results cited in previous class, the contangent complex of $\mathbf{L}_z$ is $\mathcal{O}(1) \otimes \pi^*\mathcal{L}[1]$, whence it is quasi-smooth. Therefore, since being quasi-smooth is stable under pullback, $\mathrm{W} \to \mathrm{Y} \times \mathbf{P}^1$ is quasi-smooth, whence W is quasi-smooth over $\operatorname{Spec} k$. By design, W is then a cobordism between $z^! s_*[\mathrm{Y} \to \mathrm{X}]$ and $z^!(s')_*[\mathrm{Y} \to \mathrm{X}]$. □

Theorem 2.3.4. *The quasi-smooth cobordism groups assemble into a product-preserving functor*

$$\mathbf{Corr}(\mathrm{Sch}, \mathrm{qsm}, \mathrm{prop}) \to \mathbf{Ab}_*.$$

Proof. Informally, we send a span

$$\mathrm{X} \xleftarrow{f} \mathrm{Z} \xrightarrow{g} \mathrm{Y},$$

to

$$\Omega_*^{\mathrm{qsm}}(\mathrm{X}) \xrightarrow{f^!} \Omega_{*+\dim(f)}^{\mathrm{qsm}}(\mathrm{Z}) \xrightarrow{g_*} \Omega_{*+\dim(f)}^{\mathrm{qsm}}(\mathrm{Y}).$$

□

3. Classical double point degeneration

To motivate the next definition, let us recall the notion of double-point degeneration in the classical setting. This notion also plays a prominent role in Gromov-Witten theory.

We first recall that classical naive cobordism relation: if $\mathrm{X} \in \mathrm{Sch}_k$, and $\mathrm{Y}_0, \mathrm{Y}_\infty \in \mathrm{Sm}_k$ with proper morphisms $\mathrm{Y}_0, \mathrm{Y}_\infty \to \mathrm{X}$ then $[\mathrm{Y}_0 \to \mathrm{X}]$ and $[\mathrm{Y}_\infty \to \mathrm{X}]$ in $\mathcal{M}_*^+(\mathrm{X})$ are **naively cobordant** if there exists a proper morphism $q : \mathrm{Y} \to \mathrm{X} \times \mathbf{P}^1$ such that the composite $\pi : \mathrm{W} \to \mathbf{P}^1$ is flat for which the following diagram is classically cartesian:

$$\begin{array}{ccccc} \mathrm{Y}_0 & \longrightarrow & \mathrm{Y} & \longleftarrow & \mathrm{Y}_\infty \\ \downarrow & & \downarrow{\scriptstyle q} & & \downarrow \\ \mathrm{X} & \xrightarrow{0} & \mathbf{P}^1 \times \mathrm{X} & \xleftarrow{\infty} & \mathrm{X} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbf{P}^1 & \longleftarrow & \infty \end{array}$$

We call the morphism $\mathrm{Y} \to \mathrm{X} \times \mathbf{P}^1$ a **naive cobordism**.

Example 3.0.1. Consider the morphism $\mathbf{A}^1 \hookrightarrow \mathbf{P}^1$. Since it is an open immersion it is a flat morphism. However, it is not a proper morphism, hence we do not get a strange double point relation that says (the fiber at ∞) is equivalent to

Example 3.0.2. Suppose that X is a smooth, proper scheme and consider the morphism $\mathrm{X} \to \mathbf{P}^1$ which is constant ∞. This is *not* a naive cobordism since the map is not flat.

Example 3.0.3. Consider the vanishing locus $\mathrm{X} = \mathbf{V}(xt, x^2) \subset \mathbf{P}_t^1 \times \mathbf{P}_x^1$ and consider $\mathrm{X} \to \mathbf{P}_x^1$. This is a copy of $\mathbf{P}^1$ where we have thickened the point of 0 by degree 2, i.e., it is $\operatorname{Spec} k[x]/(x^2)$. This is *not* a naive cobordism since X is not smooth.

To proceed further, let us recall that if X is a scheme, then an **effective Cartier divisor** is a closed subscheme $\mathrm{D} \hookrightarrow \mathrm{X}$ such that for each $x \in \mathrm{D}$ there is an open affine neighborhood $x \in \operatorname{Spec} \mathrm{A} = \mathrm{U} \subset \mathrm{X}$ such that $\mathrm{U} \cap \mathrm{D} = \operatorname{Spec} \mathrm{A}/f$ where f is a nonzero divisor.

Definition 3.0.4. Suppose that $x \in \mathbf{P}^1(k)$ is a rational point, then we say that a morphism $\pi : \mathrm{X} \to \mathbf{P}^1$ is a **(classical) double point degeneration at** x if:

(1) X is a smooth k-scheme of pure dimension,
(2) the morphism π is flat,
(3) and we can write the following pushout diagram of schemes

$$\pi^{-1}(x) = \mathrm{A} \cup_{\mathrm{D}} \mathrm{B} \subset \mathrm{X}.$$

where
  (a) $\mathrm{A}, \mathrm{B} \hookrightarrow \mathrm{X}$ are Cartier divisors which are furthermore smooth,
  (b) $\mathrm{D} := \mathrm{A} \cap \mathrm{B}$ and A, B intersects transversely so that D is also smooth; we call D the **double point locus**.

Remark 3.0.5. Unlike the conventions of Levine-Pandharipande, we *do not allow* A, B to be empty (though D could be if the Cartier divisors do not intersect). Hence the naive cobordism relation is *not* a a double point relatioon.

Now, suppose that $\mathrm{Y} \in \mathrm{Sm}_k$ is of pure dimension d and suppose that $\mathrm{Y} \to \mathrm{X} \times \mathbf{P}^1$ be a proper morphism for which the composition

$$\pi : \mathrm{Y} \to \mathrm{X} \times \mathbf{P}^1 \to \mathbf{P}^1$$

is a double point generation at $0 \in \mathbf{P}^1(k)$. We can extract several generators of $\mathcal{M}^+(\mathrm{X})$:

$$[\mathrm{A} \to \mathrm{X}], [\mathrm{B} \to \mathrm{X}], [\mathbf{P}(\mathcal{O}_{\mathrm{D}} \oplus \mathcal{N}_{\mathrm{D}/\mathrm{A}}) \to \mathrm{D} \to \mathrm{X}],$$

where $\mathbf{P}(\mathcal{O}_{\mathrm{D}} \oplus \mathcal{N}_{\mathrm{D}/\mathrm{A}})$ is the projective closure of the conormal sheaf of D in A; we will see later that we could have chosen to $\mathcal{N}_{\mathrm{D}/\mathrm{B}}$ instead but we postpone this to a later discussion and in the derived setting. Now, suppose that $x \in \mathbf{P}^1(k)$ is a rational point which is a regular value so that the fiber Y_x is smooth and comes equipped with a map to X defining an element

$$[\mathrm{Y}_x \to \mathrm{X}].$$

Definition 3.0.6. Let $\mathrm{X} \in \mathrm{Sch}_k$ and $\mathrm{Y} \in \mathrm{Sm}_k$ of pure dimension. Let

$$\pi : \mathrm{Y} \xrightarrow{g} \mathrm{X} \times \mathbf{P}^1 \to \mathbf{P}^1$$

be a composite where g is a proper morphism, the second arrow is the projection morphism and the composite is a double point generation at $0 \in \mathbf{P}^1(k)$. Then for any $x \in \mathbf{P}^1(k)$ which is a regular value of (so that Y_x is smooth), the **associated double point relation over** X is defined to be:

$$[\mathrm{Y}_x \to \mathrm{X}] = [\mathrm{A} \to \mathrm{X}] + [\mathrm{B} \to \mathrm{X}] - [\mathbf{P}(\mathcal{O}_{\mathrm{D}} \oplus \mathcal{N}_{\mathrm{D}/\mathrm{A}}) \to \mathrm{X}].$$

Example 3.0.7. Suppose that X is a smooth proper k-variety. Since our convention differs from Levine-Pandharipande's the constant map $\mathrm{X} \to \mathbf{P}^1$ at ∞ is also not a double point degeneration. This also does not follow their convention — as stated they only allow that one of $\mathrm{A}, \mathrm{B}, \mathrm{D}$ to be empty (though, as stated, their double point degenerations do not have to be flat). However, one can also formulate a version of double point relations where one allows any of the fibers to be empty and insist that the map to $\mathbf{P}^1$ is flat.

Example 3.0.8. Let E be an elliptic curve. Then we can degenerate E into a nodal curve E' along a family $\mathrm{Y} \to \mathbf{P}^1$ where Y is smooth (draw picture). In this situation, the double point relation reads

$$[\mathrm{E} \to \operatorname{Spec} k] = 2[\mathbf{P}^1 \to \operatorname{Spec} k] - 2[\mathbf{P}(\mathcal{O}_k) \to \operatorname{Spec} k] = 0.$$

Example 3.0.9. The reader is warned that the double point relation depends not only on g and the point x but also on the decomposition of the fiber $\pi^{-1}(0)$. So, in the previous example, we could write $\mathrm{Y}_0 = \mathrm{Y}_0 \cup_{\mathrm{Y}_0} \mathrm{Y}_0$. But we are actually safe as the double point relation reads:

$$[\mathrm{Y}_0 \to \mathrm{X}] = [\mathrm{Y}_0 \to \mathrm{X}] + [\mathrm{Y}_0 \to \mathrm{X}] - [\mathbf{P}(\mathcal{O}_{\mathrm{Y}_0}) \to \mathrm{X}](= [\mathrm{Y}_0 \to \mathrm{X}]).$$

Example 3.0.10. Recall that we have Fulton's (projective) deformation to the normal cone: for simplicity suppose that we have a closed immersion

$$\mathrm{Z} \hookrightarrow \mathrm{X}$$

where both Z and X are smooth. Then the projective deformation space is obtained as the blowup $\mathrm{Bl}_{\mathrm{Z}\times\{0\}}(\mathrm{X} \times \mathbf{P}^1)$; we write the total space as

$$\pi : \overline{\mathrm{Def}(\mathrm{Z}, \mathrm{X})} \to \mathrm{X} \times \mathbf{P}^1 \to \mathbf{P}^1$$

We have that:

(1) $\pi^{-1}(x) \cong \mathrm{X}$ for any $x \in \mathbf{P}^1 \setminus \{0\}$,
(2) $\pi^{-1}(0)$ as an X-scheme identifies with the scheme

$$\mathrm{Bl}_{\mathrm{Z}}(\mathrm{X}) \cup \mathbf{P}_{\mathrm{Z}}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}} \oplus \mathcal{O}_{\mathrm{Z}})$$

with the intersection (which is transverse) identifiying with the exceptional divisor of the blowup $\mathrm{Bl}_{\mathrm{Z}}(\mathrm{X}) \to \mathrm{X}$

$$\mathrm{Bl}_{\mathrm{Z}}(\mathrm{X}) \cap \mathbf{P}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}} \oplus \mathcal{O}_{\mathrm{Z}}) = \mathbf{P}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}});$$

we note the inclusion $\mathbf{P}(\mathcal{N}\mathrm{Z}/\mathrm{X}) \hookrightarrow \mathbf{P}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}} \oplus \mathcal{O}_{\mathrm{Z}})$ is the "hyperplane at ∞" and the normal of this immersion is what is usually known as $\mathcal{O}(1)$.

Therefore the double point relation reads:

$$[\mathrm{X} = \mathrm{X}] = [\mathrm{Bl}_{\mathrm{Z}}(\mathrm{X}) \to \mathrm{X}] + [\mathbf{P}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}} \oplus \mathcal{O}_{\mathrm{Z}}) \to (\mathrm{Z} \hookrightarrow)\mathrm{X}] - [\mathbf{P}(\mathcal{O}(1) \oplus \mathcal{O}_{\mathbf{P}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}})}) \to (\mathbf{P}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}} \to)\mathrm{X}].$$

Re-arranging terms we describe the blowup of Z at X as:

$$[\mathrm{X} = \mathrm{X}] - [\mathbf{P}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}} \oplus \mathcal{O}_{\mathrm{Z}}) \to \mathrm{X}] + [\mathbf{P}(\mathcal{O}(1) \oplus \mathcal{O}_{\mathbf{P}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}})}) \to \mathrm{X}] = [\mathrm{Bl}_{\mathrm{Z}}(\mathrm{X}) \to \mathrm{X}].$$

Example 3.0.11. The classical geometry of the situation restricts what kind of decompositions can happen over the special fiber. Consider the classical scheme

$$\mathrm{X} = \operatorname{Spec} k[x, y, z, w]/(xy, wz, xz, wy).$$

Classically, we have a bicartesian diagram

$$\begin{array}{ccc} \operatorname{Spec} k & \longrightarrow & \mathbf{A}^2 \\ \downarrow & & \downarrow \\ \mathbf{A}^2 & \longrightarrow & \mathrm{X}, \end{array}$$

expressing X as two planes meeting at a single point. This cannot be the special fiber of a classical double point degeneration: indeed if $\mathrm{Y} \to \mathbf{P}^1$ is a classical double point degeneration, then since $\mathbf{A}^2 \hookrightarrow \mathrm{Y}$ is a codimension 1 smooth divisor and X is smooth, we get that X is a dimension 3 variety. If the two copies of $\mathbf{A}^2$ meets transversally, then its dimension must be $= 2 + 2 - 3 = 1$ which is not the case.

Following Levine-Pandharinpande, if X is a classical scheme, then the **algebraic cobordism group of** X is given by

$$\mathcal{M}^*(\mathrm{X})^+/\{\text{naive cobordisms} + \text{double point degenerations}\}$$

Department of Mathematics, Harvard University, 1 Oxford St. Cambridge, MA 02138, USA
E-mail address: elmanto@math.harvard.edu
URL: https://www.eldenelmanto.com/