LECTURE 8: DERIVED DOUBLE POINT RELATIONS

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Motivated by considerations in classical algebraic geometry, we proceed to define a reasonable notion of derived double point relations and define (derived) algebraic cobordism.

1. Virtual divisors

Suppose that X is a derived scheme, a **virtual effective Cartier divisor** is a derived scheme D equipped with a quasi-smooth closed immersion $i_D : D \hookrightarrow X$ which is of virtual codimension 1. The following lemma follows from our local description of quasi-smooth morphisms:

Lemma 1.0.1. *Suppose that $i_D : D \hookrightarrow X$ is a closed immersion of derived schemes. Then the following are equivalent:*

1. *i_D is a virtual effective Cartier divisor.*
2. *i_D is Zariski-locally of the form* $\operatorname{Spec} A /\!\!/ f$.

Example 1.0.2. For any classical scheme X, an effective Cartier divisor is virtual effective Cartier divisor. The notion of a virtual effective Cartier divisor relaxes the nonzerodivisor property of f in place of taking the Koszul derived subscheme instead.

The following notion is usually attributed to Deligne.

Definition 1.0.3. A **generalized effective Cartier divisor** (or, more succinctly, an **effective generalized divisor**) on a derived scheme X is a pair $(\mathcal{L}, s)$ where $\mathcal{L}$ is a rank 1 locally free sheaf on X and $s : \mathcal{L} \to \mathcal{O}_X$ is a co-section.

Suppose that $(\mathcal{L}, s)$ is a generalized divisor. Then we can associate to it a virtual effective Cartier divisor obtained as the top horizontal arrow of the following cartesian diagram

$$\begin{array}{ccc} D & \xrightarrow{i_D} & X \\ \downarrow & & \downarrow s \\ X & \xrightarrow{0} & \mathbf{V}(\mathcal{L}^\vee). \end{array}$$

In the notation of the previous classes, it would be $D = X(s)$. By a previous computation, we get that the conormal sheaf of i_D is given by

$$\mathcal{N}_{D/X} \simeq \mathcal{L}|_D.$$

We can also associate to a virtual Cartier divisor, a generalized effective Cartier divisor via the following procedure: suppose that $i_D : D \hookrightarrow X$ is an virtual effective Cartier divisor, then we can form the following fiber sequence

$$\mathcal{F} \to \mathcal{O}_X \xrightarrow{i_D^\sharp} i_{D*}(\mathcal{O}_D).$$

We claim that $\mathcal{F}$ is locally of rank 1. Indeed, Zariski-locally on X, the sequence identifies with the fiber sequence

$$\mathcal{F} \to A \to A /\!\!/ f.$$

Following classsical notation, the sheaf $\mathcal{F}$ deserves to be called $\mathcal{O}_X(-D)$. Since it is an rank 1, its $\otimes$-inverse deserves to be called $\mathcal{O}_X(D)$. As usual, we also have equivalences:

$$\mathcal{O}_X(-D)|_D \simeq \mathbf{L}_{D/X}[-1] = \mathcal{N}_{D/X},$$

expressing the restriction of $\mathcal{O}_X(-D)|_D$ as the conormal sheaf of D in X.

Now, the functors

$$\mathrm{GDiv}, \mathrm{VDiv} : \mathrm{dSch}^{\mathrm{op}} \to \mathrm{Spc}$$

parametrizing generalized effective divisors and virtual effective Cartier divisors are in fact derived stacks, i.e., they satisfy fpqc descent. In fact the stack of generalized divisors can be described by the formula

$$\mathrm{GDiv} \simeq (\mathrm{Pic}_{/\mathcal{O}})^{\simeq}.$$

In this way, GDiv obtains the structure of a derived group whose operation is induced by the tensor product:

$$(s : \mathcal{L} \to \mathcal{O}_{\mathrm{X}}, t : \mathcal{L}' \to \mathcal{O}_{\mathrm{X}}) \mapsto (s \otimes t : \mathcal{L} \otimes \mathcal{L}' \to \mathcal{O}_{\mathrm{X}}).$$

The above constructions define transformations

$$\mathrm{GDiv} \to \mathrm{VDiv} \qquad (\mathcal{L}, s) \mapsto \mathrm{X}(s),$$

$$\mathrm{VDiv} \to \mathrm{GDiv} \qquad i_{\mathrm{D}} : \mathrm{D} \hookrightarrow \mathrm{X} \mapsto \mathcal{O}_{\mathrm{X}}(-\mathrm{D}) \to \mathcal{O}_{\mathrm{X}}$$

The following theorem is due to Rydh and Khan and is the derived version of the correspondence between effective Cartier divisors and invertible sheaves with a section.

Theorem 1.0.4. *There is a canonical equivalence of derived stacks*

$$\mathrm{GDiv} \simeq \mathrm{VDiv}.$$

Furthermore the forgetful map

$$\mathrm{VDiv} \to \mathrm{Pic} \qquad (\mathcal{L}, s) \mapsto \mathcal{L}$$

completes to a cartesian diagram

$$\begin{array}{ccc} \mathbf{A}^1 & \longrightarrow & \mathrm{Spec}\,\mathbf{Z} \\ \downarrow & & \downarrow \\ \mathrm{VDiv} & \longrightarrow & \mathrm{Pic}. \end{array}$$

Proof sketch. The vertical map $\mathrm{Spec}\,\mathbf{Z} \to \mathrm{Pic}$ classifies the trivial bundle $\mathcal{O}$. Hence the fiber can be informally described as a triple $(\mathcal{L}, s, \varphi)$ where $(\mathcal{L}, s)$ is an effective generalized divisor and $\varphi : \mathcal{O}_{\mathrm{X}} \to \mathcal{L}$ is an equivalence. We then identify the fiber with $\mathbf{A}^1$ via $(\mathcal{L}, s, \varphi) \mapsto s \circ \varphi$ □

Remark 1.0.5. Let X be a classical scheme, suppose that $s : \mathcal{L} \to \mathcal{O}_{\mathrm{X}}$ is a morphism, then we say that s is a **regular** section if the induced map $\mathcal{O}_{\mathrm{X}} \to \mathcal{L}^{\vee}$ is injective. Classically there is a natural isomorphism between effective Cartier divisors and pairs $(\mathcal{L}, s)$ where $\mathcal{L}$ is a line bundle and s is a regular section. The above theorem is a derived version of this result.

Suppose that X is a derived scheme, then we say that two virtual effective divisors $i_{\mathrm{D}}, i_{\mathrm{D}'} \in \mathrm{VDiv}(\mathrm{X})$ are **linearly equivalent** if they are identified under the map $\mathrm{VDiv}(\mathrm{X}) \to \mathrm{Pic}(\mathrm{X})$, i.e., we have an equivalence $\mathcal{O}_{\mathrm{X}}(-\mathrm{D}) \simeq \mathcal{O}_{\mathrm{X}}(-\mathrm{D}')$.

2. The derived double point relations

We now come to the main idea in this lecture. Before that we give yet another strange derived phenomenon.

Example 2.0.1. Suppose that X is a quasi-smooth derived scheme of virtual dimension d and consider the constant morphism $\pi : \mathrm{X} \to \mathbf{P}^1$ at 0. The morphism π is virtual dimension $d-1$ since it factors as $\mathrm{X} \xrightarrow{d} \mathrm{Spec}\,k \xrightarrow{-1} \mathbf{P}^1$. The derived pullback over 0 is then a derived scheme $\widetilde{\mathrm{X}}$ whose underlying classical scheme conicides with X_{cl}. However, it is of virtual dimension $d-1$ and so it is *not* X itself. This derived scheme is cobordant to 0. Here are a couple of examples of this phenomenon:

(1) consider the morphism $0 \to \mathbf{P}^1$, then the derived pullback is $\Omega_0\mathbf{A}^1 \simeq \Omega_0\mathbf{P}^1$. This is a derived scheme of virtual dimension -1 and is cobordant to 0.

(2) Say X is a classical scheme, then we have a splitting $\widetilde{\mathrm{X}} \simeq \mathrm{X} \times \mathrm{T}$. Furthermore T is just the thickened point $\Omega_0 \mathbf{A}^1$ via the cartesian diagram

$$\begin{array}{ccc} \mathrm{X} & \longleftarrow & \widetilde{\mathrm{X}} \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \longleftarrow & \Omega_0 \mathbf{A}^1 \\ {\scriptstyle 0}\downarrow & & \downarrow \\ \mathbf{P}^1 & \underset{0}{\longleftarrow} & \operatorname{Spec} k \end{array}$$

More generally, we have the following result:

Lemma 2.0.2. *Suppose that* X *is a quasi-smooth derived scheme with the following property:* $\mathrm{virt.dim}(\mathrm{X}) < \dim(\mathrm{X}_{\mathrm{cl}})$*, equipped with a splitting*

$$\mathrm{X} \simeq \mathrm{X}_{\mathrm{cl}} \times \mathrm{T}.$$

Then X *is cobordant to* 0*.*

Proof. By the dimension assumption, we get that T must be negative dimensional. Furthermore, the underlying classical scheme of T must be a point. By the classification result for derived enhancements, it must be $\Omega_0 \mathbf{A}^n$ which cobords to the empty scheme. □

Definition 2.0.3. Suppose that Y is a quasi-smooth derived scheme (over a field k). A **derived double point degeneration at** $x \in \mathbf{P}^1(k)$ is a triple

$$(\pi : \mathrm{Y} \to \mathbf{P}^1, \mathrm{A} \hookrightarrow \operatorname{Spec} k \times^h_{\mathbf{P}^1} \mathrm{Y}, \mathrm{B} \hookrightarrow \operatorname{Spec} k \times^h_{\mathbf{P}^1} \mathrm{Y})$$

such that:

(1) the composites $\mathrm{A}, \mathrm{B} \hookrightarrow \mathrm{Y}$ are virtual effective Cartier divisors,
(2) the induced map $\mathrm{A}_{\mathrm{cl}} \cup \mathrm{B}_{\mathrm{cl}} \to (\operatorname{Spec} k \times^h_{\mathbf{P}^1} \mathrm{Y})_{\mathrm{cl}}$ is an isomorphism of classical schemes.

We begin with some easy observations:

Remark 2.0.4. The underlying derived scheme of A is quasi-smooth over k since it is a composite of quasi-smooth morphisms $\mathrm{A} \hookrightarrow \mathrm{Y} \to \operatorname{Spec} k$. Ditto for B. Furthermore since $\mathrm{A} \hookrightarrow \mathrm{Y}$ is a quasi-smooth immersion, we get that $\mathrm{A} \times^h_{\mathrm{Y}} \mathrm{B} \hookrightarrow \mathrm{B}$ is also quasi-smooth, whence the composite $\mathrm{A} \times^h_{\mathrm{Y}} \mathrm{B} \hookrightarrow \mathrm{B} \to \operatorname{Spec} k$ is quasi-smooth. This is one advantage of the derived point-of-view: we do not need to impose a transversality condition. As in the classical situation we typically write

$$\mathrm{D} = \mathrm{A} \times^h_{\mathrm{Y}} \mathrm{B},$$

and call D the **derived double point locus**.

Remark 2.0.5. In the classical situation, one typically writes $\pi^{-1}(0) = \mathrm{A} \cup_{\mathrm{D}} \mathrm{B}$ and, indeed, $\pi^{-1}(0)$ is obtained by gluing A and B along their common intersection. The inclusion of classical schemes into derived ones preserves colimits, hence this presentation persists on the level of derived schemes. However, it is typically not the case (and we do not insist on it) that we have a similar decomposition for derived double point degenerations.

Remark 2.0.6. A classical double-point degeneration does define a derived double point degeneration by Example 1.0.2. We note that, since Y is smooth, the closed immersion $\mathrm{D} \hookrightarrow \mathrm{X}$ is quasi-smooth whenever D is a Cartier divisor.

Example 2.0.7. Suppose that Y is a smooth k-scheme and $\mathrm{Y} \to \mathrm{X} \times \mathbf{P}^1$ is a proper morphism and denote the composite as $\pi : \mathrm{Y} \to \mathbf{P}^1$. Then the morphism $\pi^{-1}(0) \to 0 = \operatorname{Spec} k$ is quasi-smooth morphism. In this way, the notion of a derived double-point degeneration relaxes the notion of a double-point degeneration.

We will now phrase the double point relation in derived algebraic geometry. Recall that the **normal sheaf** of a closed immersion of derived schemes $\mathrm{Z} \hookrightarrow \mathrm{X}$ is defined to be

$$\mathcal{N}_{\mathrm{Z/X}} := \mathbf{L}_{\mathrm{Z/X}}[-1].$$

We have the sheaves on D given by

$$\mathcal{N}_{\mathrm{D/A}} \oplus \mathcal{O}_{\mathrm{D}} \qquad \mathcal{N}_{\mathrm{D/B}} \oplus \mathcal{O}_{\mathrm{D}}.$$

I claim:

Lemma 2.0.8. *We have a (non-canonical isomorphism)*

$$\mathbf{P}(\mathcal{O}_{\mathrm{D}} \oplus \mathcal{N}_{\mathrm{D/A}}) \simeq \mathbf{P}(\mathcal{O}_{\mathrm{D}} \oplus \mathcal{N}_{\mathrm{D/B}})$$

over D*.*

Proof. We note that the Cartier divisor A+B is linearly equivalent to any other divisor $\pi^{-1}(t) \subset \mathrm{Y}$ which does not meet D: indeed the line bundle $\mathcal{O}(\mathrm{A}+\mathrm{B})$ is the pullback of $\mathcal{O}(1)$ on $\mathbf{P}^1$ by assumption. Therefore, $\mathcal{O}(\mathrm{A}+\mathrm{B})|_{\mathrm{D}}$ is trivial and thus so is $\mathcal{O}(-\mathrm{A}-\mathrm{B})|_{\mathrm{D}}$. From this, we get that

$$\begin{aligned}\mathcal{N}_{\mathrm{D/A}} \otimes \mathcal{N}_{\mathrm{D/B}} &\simeq \mathcal{O}_{\mathrm{A}}(-\mathrm{D})|_{\mathrm{D}} \otimes \mathcal{O}_{\mathrm{B}}(-\mathrm{D})|_{\mathrm{D}}\\ &\simeq (\mathcal{O}_{\mathrm{Y}}(-\mathrm{B}) \otimes \mathcal{O}_{\mathrm{Y}}(-\mathrm{A}))|_{\mathrm{D}}\\ &\simeq \mathcal{O}_{\mathrm{D}}.\end{aligned}$$

From which we conclude that

$$(\mathcal{O}_{\mathrm{D}} \oplus \mathcal{N}_{\mathrm{A/D}}) \otimes \mathcal{N}_{\mathrm{B/D}} \simeq (\mathcal{N}_{\mathrm{B/D}} \oplus \mathcal{O}_{\mathrm{D}}).$$

Since $\mathcal{N}_{\mathrm{B/D}}$ is free of rank 1 they induce equivalent projective bundles [1] □

We write $\mathbf{P}(\pi)$ for either of these projective bundles. It comes with a structure morphism $\mathbf{P}(\pi) \to \mathrm{D}$ which is quasi-smooth; hence the composite $\mathbf{P}(\pi) \to \mathrm{D} \to \mathrm{X}$ is quasi-smooth. Given a derived double point degeneration, the **derived double point relation** is given as follows:

Definition 2.0.9. Let $\mathrm{X} \in \mathrm{dSch}_k$ and $\mathrm{Y} \in \mathrm{QSm}_k$ of pure virtual dimension. Let

$$\pi : \mathrm{Y} \xrightarrow{g} \mathrm{X} \times \mathbf{P}^1 \to \mathbf{P}^1$$

be a composite where g is a proper morphism, the second arrow is the projection morphism and the composite is a derived double point generation at $0 \in \mathbf{P}^1(k)$. Then for any $x \in \mathbf{P}^1(k)$, the **associated double point relation over** X is defined to be:

$$[\mathrm{Y}_x \to \mathrm{X}] = [\mathrm{A} \to \mathrm{X}] + [\mathrm{B} \to \mathrm{X}] - [\mathbf{P}(\pi) \to \mathrm{X}].$$

We define

$$\Omega_*(\mathrm{X}) := \Omega_*^{\mathrm{qs}}(\mathrm{X})/\{\text{double point relations}\},$$

to be the **derived algebraic cobordism group** of X. Of course this is sort of redundant as the derived double point relation includes the derived cobordism relation. One checks that we get the pullback and pushforwards preserves the double point relation:

Theorem 2.0.10. *The derived algebraic cobordism groups assemble into a functor*

$$\Omega_* : \mathbf{Corr}(\mathrm{dSch}_k, \mathrm{qsm}, \mathrm{prop}) \to \mathrm{Ab}_*.$$

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[1]Just as in the classical case: if X is a derived scheme and $\mathcal{E}$ a locally free sheaf on X and $\mathcal{L}$ is a line bundle on X, there is a canonical X-equivalence $\varphi : \mathbf{P}(\mathcal{E}) \to \mathbf{P}(\mathcal{E} \otimes \mathcal{L})$ such that $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1) \simeq \pi^*\mathcal{L}^\vee \otimes \varphi^*(\mathcal{O}_{\mathbf{P}(\mathcal{E}\otimes\mathcal{L})}(1))$.