LECTURE 10: LAZARD'S THEOREM, QUILLEN'S THEOREM, CONSTRUCTING AN FGL I

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1. Formal group laws

Let R be a ring. What does it mean to specify a formal group law? Well a formal group law is a power series in R in two variables:

$$F(x, y) = \sum_{i,j \geqslant 0} a_{ij} x^i y^j.$$

Furthermore, it subject to certain constrains. For example since we have that $0 +_F x = x = x +_F 0$ we get that

$$a_{i0} = a_{0j} = \begin{cases} 1 & \text{if } i = j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, the commutativity relation tells us that

$$a_{ij} = a_{ji}.$$

We denote by I the ideal in $\mathbf{Z}[a_{ij}]$ generated by the relations of a formal group law. Hence to specify a formal group law F over R is equivalent to specifiying a morphism of rings

$$\varphi : \mathbf{Z}[a_{ij}]/I \to R.$$

Unpacking definitions, the formal group F is obtained by

$$F(x, y) = \sum_{i,j} \varphi(a_{ij}) x^i y^j = \varphi^* F_{\text{univ}}(x, y)$$

Let us denote by $\mathbf{L}_*$ the ring $\mathbf{Z}[a_{ij}]$; the $*$ indicates grading and we define

$$|a_{ij}| = 2(i + j - 1).$$

With this convention, and the convention that $|x| = |y| = -2$ (because we want to think of them as Chern classes of line bundles), the total degree of $F(x, y)$ is -2. Furthermore since $a_{00} = 0, a_{10} = a_{01} = 1$, we get that the ring $\mathbf{L}_*$ is nonnegatively graded. By design we get a functorial isomorphism

$$\text{Hom}(\mathbf{L}_*, R) \cong \text{FGL}(R).$$

More interestingly, we have the following very deep result:

Theorem 1.0.1 (Lazard). *The ring $\mathbf{L}_*$ is canonically isomorphic to $\mathbf{Z}[t_i]$ where $|t_i| = 2i$.*

Indeed, it is rather surprising that Lazard's ring is free polynomial! Lazard's theorem deserves its own discussion and is purely algebraic fact whose proof we will skip. But it can be translated into geometry in the following way. We have the **moduli stack of formal groups FG** which parametrizes 1-dimensional formal groups. There is a $\mathbf{G}_m$-torsor over it called the **moduli stack of formal group laws FGL**; the canonical morphism

$$\mathbf{FGL} \to \mathbf{FG}$$

forgets about coordinates on **FGL**. Lazard's theorem then tells us that there is a cover $\text{Spec}\, \mathbf{Z}[t_i] \twoheadrightarrow \mathbf{FGL}$.

The significance of Lazard's theorem for us comes as follows. By the universal property of $\mathbf{L}_*$, we obtain a canonical morphism (which obeys grading by our choice of grading on $\mathbf{L}_*$):

$$\mathbf{L}_* \to \mathrm{MU}_*.$$

Then, Quillen proved the following result twice over:

Theorem 1.0.2 (Quillen). *The morphism $\mathbf{L}_* \to \mathrm{MU}_*$ is an isomorphism.*

Quillen's theorem is also of geometric consequence: it computes the cobordism ring of almost complex manifolds as a free polynomial algebra.

Let us unpack what this theorem says about formal group laws. Suppose that $\mathcal{L}, \mathcal{L}'$ are $\mathbf{C}$-line bundles on a topological space classified by a maps

$$f, f' : \mathrm{X} \to \mathbf{CP}^\infty.$$

Then the first MU-Chern classes of $\mathcal{L}$ and $\mathcal{L}'$ lie in $\mathrm{MU}^2(\mathrm{X})$ and given by pullbacks

$$c_1(\mathcal{L}) = f^* c_1(\mathcal{O}(1)) \qquad c_1(\mathcal{L}') = f'^* c_1(\mathcal{O}(1)).$$

On the other hand we also have the tensor product of $\mathcal{L}$ and $\mathcal{L}'$ classified by maps

$$\mathrm{X} \xrightarrow{f \times f'} \mathbf{CP}^\infty \times \mathbf{CP}^\infty \xrightarrow{m} \mathbf{CP}^\infty.$$

and the associated first Chern class of the tensor product is given by

$$c_1(\mathcal{L} \otimes \mathcal{L}') = (m \circ (f \times f'))^*(c_1(\mathcal{O}(1)) = (f \times f')^* m^*(c_1(\mathcal{O}(1)) = (f \times f')^*(c_1(\mathcal{O}(1,1)).$$

Therefore we get the following relation:

$$
\begin{aligned}
c_1(\mathcal{L} \otimes \mathcal{L}') &= (f \times f')^*(c_1(\mathcal{O}(1,1))) \\
&= (f \times f')^*(c_1(\mathcal{O}(1,0) \otimes \mathcal{O}(0,1))) \\
&= (f \times f')^*(\mathrm{F}_{\mathrm{univ}}(c_1(\mathcal{O}(1,0)), c_1(\mathcal{O}(0,1))) \\
&= (f \times f')^*(\mathrm{F}_{\mathrm{univ}}(\pi_1^* c_1(\mathcal{O}(1)), \pi_2^* c_1(\mathcal{O}(1)))) \\
&= \mathrm{F}_{\mathrm{univ}}(c_1(\mathcal{L}), c_1(\mathcal{L}')),
\end{aligned}
$$

where $\pi_1, \pi_2 : \mathbf{CP}^\infty \times \mathbf{CP}^\infty \to \mathbf{CP}^\infty$ are the projection maps. We make some observations:

(1) The computation of $\otimes$ of the first Chern class of line bundles boils down to understanding $\mathcal{O}(1,1)$ versus $\pi_1^* \mathcal{O}(1)$ and $\pi_2^* \mathcal{O}(1)$.
(2) Suppose that we are at the finite level, i.e., we consider the subcomplex $\mathbf{CP}^n \times \mathbf{CP}^m \subset \mathbf{CP}^\infty \times \mathbf{CP}^\infty$, then a generic section of $\mathcal{O}(1,1)|_{\mathbf{CP}^n \times \mathbf{CP}^m}$ cuts out a **Milnor hypersurface**, while $\mathcal{O}(1)|_{\mathbf{CP}^m}$ cuts out a linear hyperplane, i.e., a copy of $\mathbf{CP}^n$.
(3) The formal group law relation then encodes the way that one can express the class of the Milnor hypersurface in terms of the hyperplanes and products thereof (since the formal group law involves terms that look like $c_1(\mathcal{O}(1))^i$ which are geometrically cutting out $\mathbf{CP}^{m-i} \hookrightarrow \mathbf{CP}^m$.

The next lemma makes (3) more precise. For now, $\mathrm{H}_{n,m}$ is defined to be the vanishing of a generic section of $\mathcal{O}(1,1)|_{\mathbf{CP}^n \times \mathbf{CP}^m}$.

Lemma 1.0.3. *Consider the formal group law* F *on* MU^* *which we write as*

$$\mathrm{F}(x, y) = \sum a_{ij} u^i v^j.$$

Then in $\mathrm{MU}^*(\mathbf{CP}^n \times \mathbf{CP}^m)$ *we can write*

$$[\mathrm{H}_{n,m} \to \mathbf{CP}^n \times \mathbf{CP}^m] = \sum_{i \geq 0}^n \sum_{j \geq 0}^m a_{ij}[\mathbf{CP}^{n-i} \times \mathbf{CP}^{m-j} \to \mathbf{CP}^n \times \mathbf{CP}^m].$$

Proof. The universal formal group law in MU^* satisfies:

$$c_1(\mathcal{O}(1,0)) +_F c_1(\mathcal{O}(0,1))) = c_1(\mathcal{O}(1,1)).$$

This equation takes place in $\mathrm{MU}^*(\mathbf{CP}^\infty \times \mathbf{CP}^\infty)$. But we have a canonical epimorphism of MU^*-modules induced by restriction:

$$\mathrm{MU}^*(\mathbf{CP}^n \times \mathbf{CP}^m) \cong \mathrm{MU}^*[x,y]/(x^{n+1}, y^{m+1}) \twoheadleftarrow \mathrm{MU}^*(\mathbf{CP}^\infty \times \mathbf{CP}^\infty) \cong \mathrm{MU}^*[[x,y]].$$

Having this, we perform the following computation in $\mathbf{CP}^n \times \mathbf{CP}^m$ (with the projection maps denoted by p_n, p_m):

$$
\begin{aligned}
[\mathrm{H}_{n,m} \to \mathbf{CP}^n \times \mathbf{CP}^m] &= c_1(\mathcal{O}(1,1)) \\
&= c_1(p_n^* \mathcal{O}_{\mathbf{CP}^n}(1) \otimes p_m^* \mathcal{O}_{\mathbf{CP}^m}(1)) \\
&= c_1(p_n^* \mathcal{O}_{\mathbf{CP}^n}(1)) +_F c_1(p_m^* \mathcal{O}_{\mathbf{CP}^m}(1)) \\
&= p_n^*(c_1(\mathcal{O}_{\mathbf{CP}^n}(1))) +_F p_m^*(c_1(\mathcal{O}_{\mathbf{CP}^m}(1))) \\
&= p_n^*(c_1(\mathcal{O}_{\mathbf{CP}^n}(1))) + p_m^*(c_1(\mathcal{O}_{\mathbf{CP}^m}(1)))) + \cdots \\
&+ a_{ij} p_m^*(c_1(\mathcal{O}_{\mathbf{CP}^m}(1)))^i \cdot p_n^*(c_1(\mathcal{O}_{\mathbf{CP}^n}(1)))^j + \cdots \\
&= \sum_{i \geqslant 0}^n \sum_{j \geqslant 0}^m a_{ij}[\mathbf{CP}^{n-i} \times \mathbf{CP}^{m-j} \to \mathbf{CP}^n \times \mathbf{CP}^m].
\end{aligned}
$$

$\square$

In the algebraic setting, we would also like to prove that $\Omega_*(\operatorname{Spec} k) \cong \mathbf{L}_*$. In particular we need to construct a formal group law on $\Omega_*(\operatorname{Spec} k)$. We will, however, not be able to prove the projective bundle formula directly. Rather our strategy is as follows:

(1) Prove that we can write the Milnor hypersurfaces as a linear combination of bilinear embeddings.
(2) From these dependencies, extract the coefficients a_{ij} in $\Omega_*(k)$.
(3) Define the power series $\mathrm{F}_{\mathrm{univ}}(x,y) := \sum a_{ij} x^i y^j$ and verify directly that this defines a formal group law.
(4) Prove that the induced morphism is an isomorphism.

2. INTERLUDE: THE PROJECTIVE BUNDLE FORMULA

There is a dream strategy which I do not know how to make work. This involves the projective bundle formula in cobordism. Let us formulate this idea in cobordism but it works for any Borel-Moore functor.

Say $\mathcal{E}$ is a locally free sheaf of rank $n+1$. Then we have the projective bundle $p_{\mathcal{E}} : \mathbf{P}_{\mathrm{X}}(\mathcal{E}) \to \mathrm{X}$ which is a projective bundle whose fibers are $\mathbf{P}^n$'s. The relative dimension of $p_{\mathcal{E}}$ is thus n and we have the pullback morphism:

$$p_{\mathcal{E}}^! : \Omega_*(\mathrm{X}) \to \Omega_{*+n}(\mathbf{P}_{\mathrm{X}}(\mathcal{E})).$$

The projective bundle $\mathbf{P}_{\mathrm{X}}(\mathcal{E})$ comes with a canonical invertible sheaf $\mathcal{O}(1)$ and we have the Chern class operators

$$\Omega_*(\mathbf{P}_{\mathrm{X}}(\mathcal{E})) \xrightarrow{\widetilde{c}_1(\mathcal{O}(1))} \Omega_{*-1}(\mathbf{P}_{\mathrm{X}}(\mathcal{E}))$$

For each $0 \leqslant j \leqslant n$ we have a morphism:

$$\varphi_j := (\widetilde{c}_1(\mathcal{O}(1)))^j \circ p_{\mathcal{E}}^! : \Omega_{*-n+j}(\mathrm{X}) \to \Omega_*(\mathbf{P}_{\mathrm{X}}(\mathcal{E})).$$

Taking the direct sum, we get

$$\Phi_{\mathcal{E}} : \Omega_{*-n}(\mathrm{X}) \oplus \Omega_{*-n+1}(\mathrm{X}) \oplus \cdots \oplus \Omega_*(\mathrm{X}) \cong \bigoplus_{j=0}^n \Omega_{*-n+j}(\mathrm{X}) \xrightarrow{\oplus \varphi_j} \Omega_*(\mathbf{P}_{\mathrm{X}}(\mathcal{E}))$$

We say that the **projective bundle formula for** $(X, \mathcal{E})$ holds in cobordism if $\Phi_\mathcal{E}$ is an isomorphism. In particular, if X is smooth so that we have the Chern class

$$\widetilde{c}_1(\mathcal{O}(1))(1_{\mathbf{P}_X(\mathcal{E})}) = c_1(\mathcal{O}(1)) \in \Omega_{-1}(\mathbf{P}_X(\mathcal{E})),$$

then, $\Omega_*(\mathbf{P}_X(\mathcal{E}))$ is a free $\Omega_*(X)$-module with generators $1, c_1(\mathcal{O}(1)), c_1(\mathcal{O}(1))^2, \cdots, c_1(\mathcal{O}(1))^n$. If we write $c_1(\mathcal{O}(1))$ as ξ, then we obtain the relation:

$$\xi^{n+1} = c_1 \xi^{n-1} - c_2 \xi^{n-2} + \cdots + (-1)^n c_n,$$

where $c_i \in \Omega_{-2i}(X)$. We then set

$$c_i(\xi) := c_i$$

to be the i-th Chern class of the vector bundle $\mathcal{E}$.

In topology, the projective bundle formula is proved via an argument using the Atiyah-Hirzerbruch spectral sequence which we do not have at our disposal in this setting. This short circuits the construction of the formal group law on cobordism. However, we will construct a putative inverse; it will be useful to denote the Chern class operator by ξ and its iterates by ξ^j.

Construction 2.0.1. *We construct maps*

$$\psi_0, \cdots, \psi_n : \Omega_*(\mathbf{P}_X(\mathcal{E})) \to \Omega_{*-n+j}(X).$$

We start by defining $\psi_0 := q_* \xi^n$. *Suppose that* $\psi_0, \cdots \psi_{m-1}$ *has been defined then we define*

$$\psi_m := q_*(\xi^{n-m} \circ (\mathrm{Id} - \sum_{j=0}^{m-1} \xi^j \circ q^! \circ \psi_j)).$$

Lemma 2.0.2. *Suppose that* X *is quasi-smooth. We get that*

$$\psi_i \circ \varphi_j = \begin{cases} \mathrm{Id} & \text{if } i = j \\ 0 & \text{Otherwise.} \end{cases}$$

In particular, $\Phi_\mathcal{E}$ *is always injective.*

Proof. For all $m \leqslant n$, define $\mathbf{P}(\mathcal{E})_X(m)$ to be the derived scheme obtained as the derived vanishing locus of a section of $\mathbf{O}(1)^{n-m}$; generically this defines the classical scheme which is a projective bundle with fibers $\mathbf{P}^m$; note that these schemes are all cobordant. We have a closed immersion over X:

$$i_m : \mathbf{P}(\mathcal{E})_X(m) \hookrightarrow \mathbf{P}(\mathcal{E})_X,$$

and a canonical morphism $q_m : \mathbf{P}(\mathcal{E})_X(m) \to X$

We claim that:

$$\xi^{n-m} \circ q^! = i_{m*} q_m^!.$$

Indeed, we have the following derived cartesian square

$$\begin{CD}
\mathbf{P}(\mathcal{E})_X(m) @>{i_m}>> \mathbf{P}_X \\
@V{i_m}VV @VV{z_m}V \\
\mathbf{P}_X @>{z_m}>> \mathbf{V}(\mathcal{O}(1)^{n-m}),
\end{CD}$$

so that the base change formula gives

$$\begin{aligned}
\xi^{n-m} \circ q^! &= z_m^! z_{m*} q^! \\
&= i_{m*} i_m^! q^! \\
&= i_{m*} q_m^!.
\end{aligned}$$

On the other hand, we also have:

$$q_*(\xi^{n-m} \circ q^!(-)) = [\mathbf{P}(\mathcal{E})_X^m \to X] \cdot (-).$$

Indeed:

$$
\begin{aligned}
q_*(\xi^{n-m} \circ q^!(-)) &= q_*(z_m^! z_{m*} q^!) \\
&= q_*(i_{m*} i_m^! q^!) \\
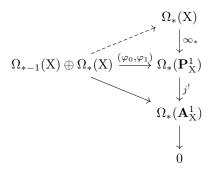&= [\mathbf{P}(\mathcal{E})_X^m \to X] \cdot (-).
\end{aligned}
$$

For the last equality to make sense, we need X to be a quasi-smooth derived scheme so that the projective bundle is indeed a cycle.

$\square$

However, it is *not obvious at all* that $\Phi_{\mathcal{E}}$ is surjective. Let us attempt to prove this for the special case of $\mathbf{P}_X^1 \to X$ and see what is at stake. Indeed, we want to prove that the map

$$
\Phi = (\varphi_0, \varphi_1) : \Omega_{*-1}(X) \oplus \Omega_*(X) \to \Omega_*(\mathbf{P}_X^1)
$$

is surjective. Consider the following commutative diagram

$$
\begin{array}{c}
\Omega_*(X) \\
\downarrow{\scriptstyle \infty_*} \\
\Omega_{*-1}(X) \oplus \Omega_*(X) \xrightarrow{(\varphi_0, \varphi_1)} \Omega_*(\mathbf{P}_X^1) \\
\downarrow{\scriptstyle j^!} \\
\Omega_*(\mathbf{A}_X^1) \\
\downarrow \\
0
\end{array}
$$

First I claim that:

$$
j^! \varphi_1 = 0.
$$

Indeed, we have the following computation in cobordism

$$
\begin{aligned}
j^! \varphi_1 &= j^! z^! z_* p^! \\
&= (zj)^! z_* p^! \\
&= 0.
\end{aligned}
$$

since we have a derived cartesian diagram (up to cobordism)

$$
\begin{array}{ccc}
\varnothing & \longrightarrow & \mathbf{P}^1 \\
\downarrow & & \downarrow{\scriptstyle z} \\
\mathbf{A}^1 & \longrightarrow & \mathbf{V}(\mathcal{O}(1)).
\end{array}
$$

Now we assume two things:

(Localization) We have a short exact sequence

$$
\Omega_*(X) \to \Omega_*(\mathbf{P}_X^1) \to \Omega_*(\mathbf{A}_X^1) \to 0.
$$

(Homotopy invariance) The canonical map $\pi : \mathbf{A}_X^1 \to X$ induces an surjection $\pi^! : \Omega_{*-1}(X) \to \Omega_*(\mathbf{A}_X^1)$.

In this situation, the dashed arrows do exist and identifies the $\Omega_*(X)$'s. On the other hand, by homotopy invariance, we get an isomorphism $\Omega_*(X) \cong \Omega_*(\mathbf{A}^1)$ via the composite of $j^! \varphi_0$. The localization and homotopy invariance property will turn out to be true for cobordism — but we will not assume them to construct the formal group laws.

3. LINEAR EMBEDDINGS

Consider the line bundle $\mathcal{O}(1)$ on $\mathbf{P}^N$. Then a **linear hypersurface** is a closed subscheme

$$\mathbf{P}^{N-1} \hookrightarrow \mathbf{P}^N$$

defined as the vanishing locus of a generic section of $\mathcal{O}(1)$. More generally an **linear embedding** is a morphism

$$\mathbf{P}^{N-j} \hookrightarrow \mathbf{P}^N$$

defined as the the vanishing locus of a generic section of $\mathcal{O}(1)^{\oplus j}$ and a **multilinear embedding** is a product of liner embeddings:

$$\prod_{}^{m} \mathbf{P}^{N_k - j_k} \hookrightarrow \prod_{}^{m} \mathbf{P}^{N_k}.$$

Since we are in derived world, it is perhaps to even defined a linear embedding as the derived zero section of a section of $\mathcal{O}(1)$ which, in our notation, is written as $\mathbf{P}^N(s)$; but note that these are derived cobordant to each other. Therefore, for each tuple:

$$J \leqslant N := (j_1 \leqslant N_1, \cdots, j_m \leqslant N_m),$$

we get a class

$$M_{J \leqslant N} = [\prod_{}^{m} \mathbf{P}^{N_k - j_k} \hookrightarrow \prod_{}^{m} \mathbf{P}^{N_k}] \in \Omega_*(\prod_{}^{m} \mathbf{P}^{N_k})$$

Lemma 3.0.1. *The classes $\{M_{J \leqslant N}\}$ are linearly independent.*

Proof. We drop the N from $J \leqslant N$. There is an obvious partial ordering on J. Suppose that

$$\alpha = \sum_{J} a_J M_J = 0,$$

but a_J is not all zero. Let $J_0 = (j_1, \cdots, j_m)$ be a minimal multi-index for which a_J is non-zero. Taking the pullback along the map (complementary to $M_{J_0 \leqslant N}$)

$$i : \prod \mathbf{P}^{j_1} \hookrightarrow \prod \mathbf{P}^{N_i}$$

gives:

$$i^!(\alpha) = \alpha_{J_0} \cdot [\prod \mathbf{P}^0 \hookrightarrow \prod \mathbf{P}^{j_i}],$$

since J_0 was assumed to be a minimal multi-index. But now, let $q : \prod \mathbf{P}^{j_i} \to \operatorname{Spec} k$ be the structure morphism, then we get that

$$q_* i^!(\alpha) = \alpha_{J_0} \neq 0.$$

Hence α could not have been zero. $\qquad \square$

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