LECTURE 9: DERIVED BLOWUPS, FORMAL GROUP LAWS

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Suppose that X is a manifold and $\mathcal{L}, \mathcal{L}' \to \mathrm{X}$ are complex line bundles. Then the first chern class of $\mathcal{L} \otimes \mathcal{L}'$ in singular cohomology admits the following relation:

$$c_1(\mathcal{L} \otimes \mathcal{L}') = c_1(\mathcal{L}) + c_1(\mathcal{L}').$$

This is *not the norm*. K-theory (say, the topological version) also admits a theory of Chern classes. The first Chern class in topological K-theory $c_1(\mathcal{L}) \in \mathrm{KU}_0(\mathrm{X})$ is defined/given by the class

$$c_1(\mathcal{L}) := 1 - [\mathcal{L}].$$

Example 0.0.1. Let us motivate why this is so. Recall that if (X, A) is a pair (where A is a closed subspace of X assuming some technical hypotheses), then we have the group $\mathrm{KU}^0(\mathrm{X}, \mathrm{X}\backslash\mathrm{A})$ of complexes of vector bundles on X which are acyclic away from A. Now, suppose that $\pi : \mathcal{E} \to \mathrm{B}$ is $\mathbf{C}$-vector bundle. The relative group

$$\mathrm{KU}^0(\mathcal{E}, \mathcal{E} \setminus 0_{\mathrm{B}})$$

is actually isomorphic to $\mathrm{KU}^0(\mathrm{B})$ via a **Thom isomorphism**

$$t_{\mathcal{E}} : \mathrm{KU}^0(\mathrm{B}) \to \widetilde{\mathrm{KU}}^0(\mathcal{E}, \mathcal{E} \setminus 0_{\mathrm{B}}) \qquad 1 \mapsto t_{\mathcal{E}}.$$

Here the tilde indicates reduced cohomology. The class $t_{\mathcal{E}}$ is represented by the Koszul complex of $\mathcal{E}$ over $\mathcal{E}$, $\{\wedge^{\bullet}\pi^*\mathcal{E}\}$; indeed this complex is acyclic away from the zero section. By definition, the Euler class of $\mathcal{E}$ is defined as

$$z^*(t_{\mathcal{E}}) = e_{\mathcal{E}} \in \mathrm{KU}^0(\mathrm{B}).$$

Notice that this is the same formula as the first Chern class in algebraic cobordism which we had discussed earlier.

Suppose that $\mathcal{L}$ is a complex line bundle, then the above procedure leads to the complex

$$0 \to \pi^*\mathcal{L} \to \underline{\mathbf{C}} \to 0$$

which pulls back to the complex $0 \to \mathcal{L} \to \underline{\mathbf{C}} \to 0$. As a class in $\mathrm{KU}^0(\mathrm{B})$ this is represented by $1 - [\mathcal{L}]$.

By choice, the first Chern class defines coordinates for the projective bundle formula:

$$\mathrm{KU}^0(\mathrm{C}\mathbf{P}^1_{\mathrm{X}}) \to \mathrm{KU}^0(\mathrm{X})\{\mathcal{O}_{\mathrm{X}}\} \oplus \mathrm{KU}^0(\mathrm{X})\{c_1(\mathcal{O}(1))(= 1 - [\mathcal{L}])\}.$$

The Chern class relation in topological K-theory is then given by

$$c_1(\mathcal{L}) + c_1(\mathcal{L}') - c_1(\mathcal{L}')c_1(\mathcal{L}) = c_1(\mathcal{L} \otimes \mathcal{L}').$$

In a moment, we will see that this leads to the notion of a formal group law on a cohomology theory, but the slogan that one keeps in mind that is the there is a "junk term" which prevents c_1 of line bundles from converting $\otimes$ to $+$. The **extended double point relation** expresses this discrepancy in algebraic cobordism and is, in fact, the "universal discrepancy." This idea will lead to formal group laws which appears in topology and will appear in the theory of algebraic cobordism.

1. Derived blowups

In order to phrase the derived analog of the blowup relation, we need to speak of the derived blowups. This is the work of Kerz-Strunk-Tamme and Rydh-Khan.

Definition 1.0.1. Suppose that $\mathrm{Z} \hookrightarrow \mathrm{X}$ is a quasi-smooth closed immersion of derived schemes. For any derived scheme S equipped with a map $f : \mathrm{S} \to \mathrm{X}$, a **virtual Cartier divisor lying over** (X, Z) is the data of a commutative diagram

$$\begin{array}{ccc} \mathrm{D} & \xrightarrow{i_{\mathrm{D}}} & \mathrm{S} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ \mathrm{Z} & \longrightarrow & \mathrm{X}. \end{array}$$

satisfying the following conditions:

(1) i_{D} is a virtual Cartier divisor,
(2) the underlying square of classical schemes is Cartesian (note that the square is not, in general, demanded to be cartesian — $\mathrm{Z} \hookrightarrow \mathrm{X}$ is not virtual codimension 1),
(3) the canonical map $g^*\mathcal{N}_{\mathrm{Z}/\mathrm{X}} \to \mathcal{N}_{\mathrm{D}/\mathrm{S}}(\simeq \mathcal{O}(-\mathrm{D})|_{\mathrm{D}})$ is surjective on π_0.

Example 1.0.2. Suppose that $\mathrm{Z} \hookrightarrow \mathrm{X}$ is a regular immersion. Then the blowup square (which is classically cartesian)

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathrm{Bl}_{\mathrm{Z}}\mathrm{X} \\ \downarrow & & \downarrow \\ \mathrm{Z} & \longrightarrow & \mathrm{X}. \end{array}$$

is an example of such a diagram. We warn the reader, however, that $\mathcal{E} \to \mathrm{Bl}_{\mathrm{Z}}(\mathrm{X}) \times^h_{\mathrm{X}} \mathrm{Z}$ is not, in general, an isomorphism so a blowup square is an example of a classically cartesian diagram which is not derived cartesian.

Consider the derived pullback $\mathrm{S} \times^h_{\mathrm{X}} \mathrm{Z} =: \mathrm{S}_{\mathrm{Z}}$. Then there is a canonical morphism

$$e : \mathrm{D} \to \mathrm{S}_{\mathrm{Z}}$$

which, by assumption (2), is an isomorphism on underlying classical schemes. We note that $\mathrm{S}_{\mathrm{Z}} \hookrightarrow \mathrm{S}$ has the same virtual codimension as $\mathrm{Z} \hookrightarrow \mathrm{X}$. We have a canonical map

$$e^\sharp : \mathcal{O}_{\mathrm{S}_{\mathrm{Z}}} \to e_*(\mathcal{O}_{\mathrm{D}})$$

Remark 1.0.3. Conditions (2)+(3) is equivalent to saying that $\pi_0(e^\sharp)$ and $\pi_1(e^\sharp)$ are isomorphisms. This will be left as an exercise to the reader to unpack.

Define a functor

$$\mathrm{Bl}_{\mathrm{Z}}\mathrm{X} : \mathrm{dSch}^{\mathrm{op}}_{\mathrm{X}} \to \mathrm{Spc}$$

which sends an X-scheme T to the ∞-groupoid of virtual Cartier divisor lying over (X, Z). These conditions are clearly fpqc local. Furthermore we have the following theorem:

Theorem 1.0.4 (Rydh-Khan)**.** *Let $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ be a quasi-smooth closed immersion of derived schemes. Then*

(1) *The* X*-stack* $\mathrm{Bl}_{\mathrm{Z}}\mathrm{X}$ *is a derived scheme; we call* $\mathrm{Bl}_{\mathrm{Z}}\mathrm{X}$ *the* **derived blowup** *of* Z *along* X.
(2) *The formation of derived blowups commutes with derived base change.*
(3) *The formation of derived blowups is functorial along closed immersions: for ay quasi-smooth closed immersion $i : \mathrm{X} \hookrightarrow \mathrm{Y}$ there is a canonical quasi-smooth closed immersion*

$$\mathrm{Bl}_{\mathrm{Z}}(\mathrm{X}) \hookrightarrow \mathrm{Bl}_{i(\mathrm{Z})}(\mathrm{Y}).$$

(4) *There is a canonical closed immersion*

$$\mathbf{P}(\mathcal{N}_{Z/X}) \hookrightarrow \mathrm{Bl}_Z X,$$

which exhibits $\mathbf{P}(\mathcal{N}_{Z/X})$ *as the universal virtual Cartier divisor lying over* (X, Z)*; we denote this virtual divisor by* $\mathcal{E}$ *and call this the* **universal Cartier divisor**

(5) *The morphism* $\mathrm{Bl}_Z X \to X$ *is proper and quasi-smooth and is an isomorphism away from* Z.

(6) *If* Z, X *are classical so that* i *is a regular immersion of classical schemes, then the derived blowup coincides with the usual blowup.*

(7) *Suppose that* A *is a classical ring and* $f_1, \cdots, f_r$ *which generates an ideal* I *of* A*, then the derived blowup fits into the following derived cartesian square*

$$\begin{array}{ccc} \mathrm{Bl}_{Z^{\mathrm{der}}} X & \longrightarrow & \mathrm{Bl}_0 \mathbf{A}^n \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, A & \xrightarrow{f} & \mathbf{A}^n, \end{array}$$

and the universal Cartier divisor fits into a derived cartesian square

$$\begin{array}{ccc} \mathcal{E} & \hookrightarrow & \mathrm{Bl}_{Z^{\mathrm{der}}} X \\ \downarrow & & \downarrow \\ \mathbf{P}^{n-1} & \hookrightarrow & \mathrm{Bl}_{\{0\}} \mathbf{A}^n. \end{array}$$

Proof Sketch. In classical algebraic geometry, recall that the blowup of 0 along $\mathbf{A}^2$ (which is the simplest example) can be described in the following manner:

$$\begin{array}{ccc} \mathrm{Bl}_0^{\mathrm{cl}}(\mathbf{A}^2) & \hookrightarrow & \mathbf{A}^2 \times \mathbf{P}^1 \\ \downarrow & \swarrow & \\ \mathbf{A}^2 & & \end{array} \qquad \{((T_1, T_2), [U : V]) | T_1 V - U T_2\}$$

.

This scheme can be described in two charts:

($U \neq 0$) In this chart $T_1 V = T_2$ so this is the subscheme of $\mathbf{A}^1 = \mathrm{Spec}\, \mathbf{Z}[T_1, T_2, V]$ given by $V(T_1 V - T_2)$ and we regard it as a $\mathbf{Z}[T_1, T_2]$-algebra generated by V and subject to one relation $T_1 V = T_2$. Obviously $T_1 V - T_2$ is not a zero divisor.

($V \neq 0$) In this chart, we are looking at a $\mathbf{Z}[T_1, T_2]$-algebra generated by U and subject to one relation $T_1 = T_2 U$ by a similar argument.

In both cases, we note that the classical blowup $\mathrm{Bl}_0^{\mathrm{cl}}(\mathbf{A}^2)$ is covered by two affine schemes which are principal closed subschemes of $\mathbf{A}^3$; the enhancements of these subschemes are trivial since they are cut out by nonzero divisors. Generalizing this situation, the classical blowup of $\mathrm{Bl}_0^{\mathrm{cl}}(\mathbf{A}^n)$ has charts given by

$$V(\{T_k X_r - T_r\}_{r \neq k}) \hookrightarrow \mathbf{A}^n \times \mathbf{A}^{n-1},$$

and we have an equivalence $V(\{T_k X_r - T_r\}_{r \neq k}) \simeq V^{\mathrm{der}}(\{T_k X_r - T_r\}_{r \neq k})$ since $\{T_k X_r - T_r\}$ forms a regular sequence. This suggests that:

- the classical blowup is equivalent to the derived blowup when we blowup $\mathbf{A}^n$ at $\{0\}$.

Indeed, one can prove this result as a by-product of trying to prove that $\mathrm{Bl}_{\{}0\}\mathbf{A}^n$ is represented by a derived scheme: the functor has a Zariski cover by these $V(\{T_k X_r - T_r\}_{r \neq k})$'s.

In fact, one can prove that derived blowups coincide with the classical blowup whenever the center is regularly immersed (which is one of the assertions). To proceed more generally, we note that the formation of derived blowups satisfy Zariski descent so it suffices to construct the derived blowup of $\mathrm{Spec}\, A /\!\!/ (f_1, \cdots, f_n) \to \mathrm{Spec}\, A$. Furthermore, the construction of the derived blowup commutes with base change (by inspecting the functors they represent). Whatever the derived blowup is, it must then fit into the following derived cartesian square

$$\begin{array}{ccc} \mathrm{Bl}_{\mathrm{Z}}(\mathrm{X}) & \longrightarrow & \mathrm{Bl}_0\mathbf{A}^n \\ \downarrow & & \downarrow \\ \mathrm{X} & \xrightarrow{(f_1,\cdots,f_n)} & \mathbf{A}^n. \end{array}$$

□

Remark 1.0.5. Suppose that $\mathrm{Z} \hookrightarrow \mathrm{Y} \hookrightarrow \mathrm{X}$ is a sequence of quasi-smooth closed immersions, then we have a commutative diagram

$$\begin{array}{ccc} \mathbf{P}(\mathcal{N}_{\mathrm{Z}/\mathrm{Y}}) & \longrightarrow & \mathrm{Bl}_{\mathrm{Z}}(\mathrm{Y}) \\ \downarrow & & \downarrow \\ \mathrm{Z} & & \mathrm{Y} \\ \downarrow = & & \downarrow \\ \mathrm{Z} & \longrightarrow & \mathrm{X}, \end{array}$$

whence we get a morphism

$$\mathrm{Bl}_{\mathrm{Z}}(\mathrm{X}) \hookrightarrow \mathrm{Bl}_{i(\mathrm{Z})}(\mathrm{Y}).$$

as in Theorem 1.0.4.3 whcih is furthermore a quasi-smooth closed immersion. This morphism is called the **strict transform** of $\mathrm{Y} \hookrightarrow \mathrm{X}$.

Remark 1.0.6. Suppose that $\mathrm{Z} \hookrightarrow \mathrm{X}$ is a closed immersion of classical schemes, not necessarily quasi-smooth. Then we may form the classical blowup $\mathrm{Bl}^{\mathrm{cl}}_{\mathrm{Z}}\mathrm{X}$, which admits a canonical closed immersion (not necessarily an isomorphism) $\mathrm{Bl}^{\mathrm{cl}}_{\mathrm{Z}}\mathrm{X} \hookrightarrow (\mathrm{Bl}_{\mathrm{Z}^{\mathrm{der}}}(\mathrm{X}))_{\mathrm{cl}}$. More generally, the classical locus of $\mathrm{Bl}_{\mathrm{Z}}\mathrm{X}$ is equivalent to

$$\mathrm{Proj}^{\mathrm{cl}}_{\mathrm{X}_{\mathrm{cl}}}(\mathrm{Sym}^*\pi_0(\mathcal{I})),$$

where $\mathcal{I}$ is the fiber of $\mathcal{O}_{\mathrm{X}} \to i_*(\mathcal{O}_{\mathrm{Z}})$.

Using Theorem 1.0.4 the same formula gives us a derived deformation to the normal cone: $\overline{\mathrm{Def}(\mathrm{Z},\mathrm{X})} \to \mathrm{X} \times \mathbf{P}^1 \to \mathbf{P}^1$ whose fiber over 0 contains the immersions

$$\mathbf{P}(\mathbf{N}_{\mathrm{Z}/\mathrm{X}} \oplus \mathcal{O}) \hookrightarrow \overline{\mathrm{Def}(\mathrm{Z},\mathrm{X})} \qquad \mathrm{Bl}_{\mathrm{Z}}(\mathrm{X}) \hookrightarrow \overline{\mathrm{Def}(\mathrm{Z},\mathrm{X})},$$

with the derived pullback being $\mathbf{P}(\mathbf{N}_{\mathrm{Z}/\mathrm{X}})$. This gives us the derived double-point relation:

$$[\mathrm{X} = \mathrm{X}] = [\mathrm{Bl}_{\mathrm{Z}}(\mathrm{X}) \to \mathrm{X}] + [\mathbf{P}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}} \oplus \mathcal{O}_{\mathrm{Z}}) \to (\mathrm{Z} \hookrightarrow)\mathrm{X}] - [\mathbf{P}(\mathcal{O}(1) \oplus \mathcal{O}_{\mathbf{P}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}})}) \to (\mathbf{P}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}}) \to)\mathrm{X}].$$

2. Formal group laws

The goal of the next few lectures is to construct a formal group law over the graded $\Omega_*(k)$. Let us begin with a topological digression. Suppose that E is a cohomology theory valued in rings, then we can associate to it $\widetilde{\mathrm{E}}^*$, the reduced cohomology theory valued in rings. In particular, this means that $\widetilde{\mathrm{E}}^*$ takes in a pointed CW-complex and spits out a graded ring. A **complex orientation** on $\widetilde{\mathrm{E}}^*$ is essentially the following data: a collection

$$\{u_{\mathcal{E}} \in \widetilde{\mathrm{E}}^{2\mathrm{rank}_{\mathbf{C}}(\mathcal{E})}(\mathrm{Th}_{\mathrm{X}}(\mathcal{E}))\}_{\mathcal{E} \in \mathrm{Vect}_{\mathbf{C}}(\mathrm{X})}.$$

Here, $\mathrm{Th}_{\mathrm{X}}(\mathcal{E})$ the **Thom complex** of the bundle $\mathcal{E}$ is the CW complex with the following homotopy type:

$$\mathrm{Th}_{\mathrm{X}}(\mathcal{E}) \simeq \frac{\mathbf{P}(\mathcal{E} \oplus \mathcal{O})}{\mathbf{P}(\mathcal{E})}.$$

These classes are subject to the following requirements:

(1) For each $x \in \mathrm{X}$, $u_{\mathcal{E}} \in \widetilde{\mathrm{E}}^{2\mathrm{rank}_{\mathbf{C}}(\mathcal{E})}(\mathrm{Th}_{\mathrm{X}}(\mathcal{E}))$ is sent under the composite
$$\widetilde{\mathrm{E}}^{2\mathrm{rank}_{\mathbf{C}}(\mathcal{E})}(\mathrm{Th}_{\mathrm{X}}(\mathcal{E})) \xrightarrow{i_x^*} \widetilde{\mathrm{E}}^{2\mathrm{rank}_{\mathbf{C}}(\mathcal{E})}(\mathrm{Th}_x(\mathcal{E}_x)) \xrightarrow{\cong} \widetilde{\mathrm{E}}^{2\mathrm{rank}_{\mathbf{C}}(\mathcal{E})}(\mathrm{S}^{2\mathrm{rank}_{\mathbf{C}}(\mathcal{E})}) \cong \mathrm{E}^0(\star),$$
to the element $1 \in \mathrm{E}^0(\star)$.
(2) The classes are stable under pullbacks.
(3) The classes are multiplicative: $u_{\mathcal{E}} \cdot u_{\mathcal{E}'} = u_{\mathcal{E}\oplus\mathcal{E}'}$.

In fact, a complex orientation is determined by much less data. Consider $\mathbf{CP}^\infty$ — which is a homotopy type characterized as the classifying space for $\mathbf{C}$-line bundles. There is the universal solution $\mathcal{O}(1)$ on $\mathbf{CP}_\infty$ and hence we have the Thom complex $\mathrm{Th}_{\mathbf{CP}^\infty}(\mathcal{O}(1))$. If E admits a complex orientation, then we can pull back $t_{\mathcal{O}(1)} \in \widetilde{\mathrm{E}}^2(\mathrm{Th}_{\mathbf{CP}^\infty}(\mathcal{O}(1)))$ along the zero section to a class $z^*t_{\mathcal{O}(1)} = c_1(\mathcal{L}) \in \widetilde{\mathrm{E}}^2(\mathbf{CP}^\infty)$. This class has the property that if we restrict along the map $\mathbf{CP}^1 \hookrightarrow \mathbf{CP}^\infty$ it gets mapped to 1 under the isomorphism $\widetilde{\mathrm{E}}^2(\mathbf{CP}^1) \cong \widetilde{\mathrm{E}}^2(\mathrm{S}^2) \cong \mathrm{E}^0(*)$.

Proposition 2.0.1. *Any class $x \in \widetilde{\mathrm{E}}^2(\mathbf{CP}^\infty)$ restricting to 1 under the composite*
$$\widetilde{\mathrm{E}}^2(\mathbf{CP}^\infty) \to \widetilde{\mathrm{E}}^2(\mathbf{CP}^1) \cong \mathrm{E}^0(*),$$
extends to a complex orientation in a unique way.

A choice of a class in $\widetilde{\mathrm{E}}^2(\mathbf{CP}^\infty)$ restricting to 1 as above, will be denoted. by $c_1^{\mathrm{E}}(\mathcal{O}(1))$. This is the first Chern class of the universal bundle. For convenience we might also denote this class by x_{E}. This leads to knowledge of what the first Chern class of any line bundle is: given a $\mathbf{C}$-line bundle on a topological space X we get a classifying map $\mathrm{X} \to \mathbf{CP}^\infty$ and pulling back x_{E} gives us $c_1^{\mathrm{E}}(\mathcal{L}) \in \mathrm{E}^2(\mathrm{X})$.

This more succinct data leads to more robust properties on $\widetilde{\mathrm{E}}$. For each $n \geqslant 1$ there is then a morphism
$$\mathrm{E}^*[x_{\mathrm{E}}] \to \mathrm{E}^*(\mathbf{CP}^n)$$
such that the image of x satisfies $x^{n+1} = 0$. The resulting map
$$\mathrm{E}^*[x_{\mathrm{E}}]/(x_{\mathrm{E}}^{n+1}) \to \mathrm{E}^*(\mathbf{CP}^n),$$
turns out to be an isomorphism; this is the **projective bundle formula**. There is even a 2-variable version:
$$\mathrm{E}^*[[x_{\mathrm{E}}, y_{\mathrm{E}}]] \cong \mathrm{E}^*(\mathbf{CP}^\infty \times \mathbf{CP}^\infty).$$
Taking the limit as $n \to \infty$ we obtain an isomorphism
$$\mathrm{E}^*(\mathbf{CP}^\infty) \cong \mathrm{E}^*[[x_{\mathrm{E}}]].$$
Now consider the map
$$\mathbf{CP}^\infty \times \mathbf{CP}^\infty \to \mathbf{CP}^\infty$$
which classifies tensoring of line bundles. This gives rise to a map on cohomology
$$\mathrm{E}^*[[x_{\mathrm{E}}]] \to \mathrm{E}^*[[x_{\mathrm{E}}, y_{\mathrm{E}}]].$$
The image of x is a power series in two variables and by construction
$$c_1^{\mathrm{E}}(\mathcal{L}_1 \otimes \mathcal{L}_2) = \mathrm{F}(c_1^{\mathrm{E}}(\mathcal{L}_1), c_2^{\mathrm{E}}(\mathcal{L}_2)).$$

Definition 2.0.2. A **formal group law** over a ring R is a power series
$$\mathrm{F}(x, y) \in \mathrm{R}[[x, y]],$$
which we also denote by $x +_{\mathrm{F}} y$, satisfiying
(1) $x +_{\mathrm{F}} y = y +_{\mathrm{F}} x$,
(2) $x +_{\mathrm{F}} 0 = 0 +_{\mathrm{F}} x = x$
(3) $(x +_{\mathrm{F}} y) +_{\mathrm{F}} z = x +_{\mathrm{F}} (y +_{\mathrm{F}} z)$.

From the above discussion and properties of tensoring of line bundles, any complex oriented theory gives rise to a formal group law.

Example 2.0.3. Here are some basic examples of formal group laws:

(1) the **additive formal group law** is given by

$$\mathbf{G}_a(x,y) = x + y.$$

(2) The **multiplicative formal group law** is given by

$$\mathbf{G}_m(x,y) = x + y - xy.$$

(3) Let E be a relative elliptic curve over $\mathrm{Spec}\,\mathrm{R}$ which comes equipped with the identity section $e : \mathrm{Spec}\,\mathrm{R} \to \mathrm{E}$. Formally complete along the identity section yields an abelian group in formal schemes which is relative dimension 1. Étale-locally on the source we can find a coordinate, whence we can produce an isomorphism between the formal completion and $\mathrm{Spf}\,\mathrm{R}[[x]]$. The group structure unpacks to a formal group law on R.

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