ALGEBRAIC K-THEORY AND THE UNSTABLE HOMOLOGY OF GENERAL LINEAR GROUPS

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Abstract. The stable homology of general linear groups over a field is well-known to be closely related to its algebraic K-theory. I will discuss joint work with Soren Galatius and Oscar Randal-Williams which investigates the unstable homology of general linear groups. We will find it is closely related to the Milnor K-theory, by constructing a presentation of the disjoint union of $BGL_n(\mathbb{F})$ as an E_∞-algebra.

This is a talk on joint work with Søren Galatius and Oscar Randal-Williams, [GKRW20]. This is the fourth paper in the series, the earlier three being [GKRW18a, GKRW19, GKRW18b]. The first discusses foundations, and the other two are applications to mapping class groups of surfaces and general linear groups of finite fields. In this talk we discuss applications to general linear groups of infinite groups; the main difference between finite and infinite fields is the Nesterenko–Suslin property discussed below.

Convention 0.1. During this talk, I will assume for the sake of simplicity that all rings A are commutative and have the property that finitely-generated projective modules are free. Soon, I will even assume A is an infinite field.

Convention 0.2. All functors are derived; either one works with ∞-categories, or I'm asserting the existence of appropriate model category structures.

1. Homology of general linear groups

We all know the definition of K_0 of a commutative ring A. The set of isomorphism classes $[P]$ of finitely generated projective A-modules forms an abelian monoid under direct sum. We then take its group completion

$$K_0(A) \coloneqq \frac{\text{free abelian group on } [P]}{[P]+[Q]-[P\oplus Q]}.$$

Example 1.1. Under Convention 0.1, the rank induces an isomorphism $K_0(A) \to \mathbb{Z}$.

A homotopical instantation of the group completion leads to one of the definitions of the higher algebraic K-theory groups; we replace the isomorphism classes of finitely generated projective A-modules by the "moduli space" of such modules:

$$\bigsqcup_{n\geq 0} B\mathrm{GL}_n(A).$$

The block sum homomorphism $\mathrm{GL}_n(A)\times \mathrm{GL}_m(A) \to \mathrm{GL}_{n+m}(A)$ induces a map on classifying spaces which makes this space into a topological monoid, and we define the algebraic K-theory

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space as

$$\Omega^\infty K(A) \coloneqq \Omega B\left(\bigsqcup_{n\geq 0} B\mathrm{GL}_n(A)\right).$$

This is Segal's construction of the group completion, the based loop space of the bar construction.

Definition 1.2. The *algebraic K-theory groups of A* are

$$K_i(A) \coloneqq \pi_i(\Omega^\infty K(A)).$$

Since it is a loop space, all path components are equivalent and we don't need to worry about basepoints. As all loop spaces are simple, there is a strong relationship between their homotopy groups and homology groups. We can understand $\Omega^\infty K(A)$ in terms of $H_*(B\mathrm{GL}_n(A))$ using the *group completion theorem* of McDuff–Segal [MS76]: the inclusion homomorphism $\mathrm{GL}_n(A) \to \mathrm{GL}_{n+1}(A)$ in the top-left corner induces maps

$$\sigma\colon B\mathrm{GL}_n(A) \to B\mathrm{GL}_{n+1}(A),$$

which assemble to a map $\Omega^\infty K(A) \to \Omega^\infty K(A)$.

Proposition 1.3 (Group completion theorem)**.** *We have that*

$$H_*(\Omega^\infty K(A)) = \operatorname{colim} H_*\left(\bigsqcup_{n\geq 0} B\mathrm{GL}_n(A) \xrightarrow{\sigma} \bigsqcup_{n\geq 0} B\mathrm{GL}_n(A) \xrightarrow{\sigma} \cdots\right).$$

In particular, the homology of a path component $\Omega^\infty_0 K(A)$ is given by the *stable homology*

$$\operatorname{colim}_{n\to\infty} H_*(B\mathrm{GL}_n(A)).$$

Thus, understanding the homology of the groups $B\mathrm{GL}_n(A)$ and the maps between them is one way to obtain information about $K_*(A)$.

Remark 1.4. As far as I know, this has only been a successfull strategy for degrees $* \leq 3$, for finite fields [Qui72], or for rings of integers [Bor74].

It may seem that implementing this strategy requires the computation of $H_*(B\mathrm{GL}_n(A))$ for all sufficiently large n, but this is not the case due to *homological stability* results. The easiest example can be worked out by hand:

Example 1.5. We have that $H_1(B\mathrm{GL}_n(A)) = \mathrm{GL}_n(A)^{\mathrm{ab}}$. The determinant gives a homomorphism $\det\colon \mathrm{GL}_n(A) \to A^\times$ which factors over the abelianization because the target is abelian. If $A = \mathbb{F}$ is a field, then this is an isomorphism for $n \geq 3$ (actually $n \geq 2$ when you assume $\mathbb{F} \neq \mathbb{F}_2$) because $\mathrm{SL}_n(\mathbb{F})$ is perfect for $n \geq 3$. Thus $K_1(\mathbb{F}) = \mathbb{F}^\times$.

Indeed, for many rings A, the map

$$H_*(B\mathrm{GL}_{n-1}(A)) \to H_*(B\mathrm{GL}_n(A))$$

is an isomorphism for $*$ in a range tending to infinity with n; the explicit bounds depend on A (or rather, the *stable rank* of A). It is better, from the point of view of stating the sharpest results and proving them, to phrase homological stability results in terms of the relative homology groups of σ. The result I want to focus on a homological stability result due to Nesterenko and Suslin [NS89]. This theorem is sharp, which is seen by determining the next relative homology groups:

Theorem 1.6 (Nesterenko–Suslin)**.** *If A is an infinite field, then*

$$H_*(\mathrm{GL}_n(\mathbb{F}), \mathrm{GL}_{n-1}(\mathbb{F})) = 0 \qquad \text{for } * < n.$$
$$H_n(\mathrm{GL}_n(\mathbb{F}), \mathrm{GL}_{n-1}(\mathbb{F})) = K_n^M(\mathbb{F}).$$

Definition 1.7. $K_*^M(\mathbb{F})$ is the *Milnor K-theory.* This is graded ring generated by symbols $\{a\}$ for $a \in \mathbb{F}^\times$ modulo the relation that $\{a\} \cdot \{b\} = 0$ if $a + b = 1$.

That is, the first obstruction to further homological stability is

$$\bigoplus_{n \geq 0} H_n(\mathrm{GL}_n(\mathbb{F}), \mathrm{GL}_{n-1}(\mathbb{F})) \cong K_*^M(\mathbb{F}).$$

Remark 1.8. Its appearance here is different from the way that Milnor K-theory usually appears: tensor product makes $K_*(\mathbb{F})$ into a graded ring, and $K_1(\mathbb{F}) = \mathbb{F}^\times$ generates a subring isomorphism to $K_*^M(\mathbb{F})$. Nesterenko and Suslin also proved that the maps

$$\begin{tikzcd}
K_*^M(\mathbb{F}) \arrow[r] \arrow[d, dashed] & K_*(\mathbb{F}) \arrow[r, "\mathrm{Hur}"] & \operatorname{colim}_{n\to\infty} H_*(B\mathrm{GL}_n(\mathbb{F})) \arrow[d, "\cong"] \\
K_n^M(\mathbb{F}) & H_n(\mathrm{GL}_n(\mathbb{F}), \mathrm{GL}_{n-1}(\mathbb{F})) \arrow[l] & H_n(B\mathrm{GL}_n(\mathbb{F})) \arrow[l]
\end{tikzcd}$$

is multiplication by $(n-1)!$. Thus after inverting sufficiently many primes $K_n^M(\mathbb{F})$ splits off $K_n(\mathbb{F})$.

In this talk, I want to give an alternative proof of Theorem 1.6 and use the same technique to give some information about the next line of *unstable groups*

$$\bigoplus_{n \geq 0} H_{n+1}(\mathrm{GL}_n(\mathbb{F}), \mathrm{GL}_{n-1}(\mathbb{F})).$$

However, the route to this result will be leasurely and along to way we will make some stops to admire the sights.

2. E_k-homology and cellular E_k-algebras

The strategy will be to give a "presentation" of $\bigsqcup_{n\geq 1} B\mathrm{GL}_n(\mathbb{F})$ as a non-unital CW-E_∞-algebra, obtaining a bound on the number of E_∞-cells through the study of certain buildings. The facts about E_k-algebras in this section can be found in [GKRW18a].

2.1. **General linear groups as E_∞-algebras.** The space $\bigsqcup_{n\geq 1} B\mathrm{GL}_n(\mathbb{F})$ is not just a topological monoid, but it is in fact an E_∞-algebra. That is, it is commutative in a homotopy-coherent sense.

Example 2.1. To see it is homotopy commutative recall that conjugation by a group element is homotopic to the identity on a classifying space BG, and the image of

$$\mathrm{GL}_n(\mathbb{F}) \times \mathrm{GL}_m(\mathbb{F}) \to \mathrm{GL}_{n+m}(\mathbb{F})$$

differs from that of

$$\mathrm{GL}_m(\mathbb{F}) \times \mathrm{GL}_n(\mathbb{F}) \to \mathrm{GL}_{n+m}(\mathbb{F})$$

by conjugation with
$$\begin{bmatrix} 0 & \mathrm{id}_m \\ \mathrm{id}_n & 0 \end{bmatrix} \in \mathrm{GL}_{n+m}(\mathbb{F}).$$

To see it is an E_∞-algebra, it is helpful to work more generally. Let $i\mathsf{P}(A)$ be the category of finitely generated projective A-modules and isomorphisms. This is a symmetric monoidal category under direct sum. Then the category $\mathsf{sSet}^{i\mathsf{P}(A)}$ of functors $i\mathsf{P}(A) \to \mathsf{sSet}$ inherits a symmetric monoidal structure from Day convolution: take the left Kan extension in

$$\begin{tikzcd} i\mathsf{P}(A) \times i\mathsf{P}(A) \arrow[r, "F \times G"] \arrow[d, "\oplus"] & \mathsf{sSet} \times \mathsf{sSet} \arrow[r, "\times"] & \mathsf{sSet} \\ i\mathsf{P}(A). \arrow[urr, dashed] & & \end{tikzcd}$$

Given this symmetric monoidal structure on $\mathsf{sSet}^{i\mathsf{P}(A)}$, we can make sense of non-unital E_∞-algebras in this category. There is a canonical such algebra with a unique multiplication: the object $\underline{*}_{>0}$ which is $\varnothing$ on zero objects and $*$ otherwise (which is in fact strictly commutative).

To construct $\bigsqcup_{n\geq 1} B\mathrm{GL}_n(\mathbb{F})$ as an E_∞-algebra from this when $A = \mathbb{F}$, we use that dimension gives a symmetric monoidal functor $r\colon (i\mathsf{P}(A), \oplus) \to (\mathbb{N}, +)$. The restriction functor $r^*\colon \mathsf{sSet}^{\mathbb{N}} \to \mathsf{sSet}^{i\mathsf{P}(\mathbb{F})}$ has a (derived) left adjoint[1] r_* which sends E_∞-algebras to E_∞-algebras: objectwise this is given by taking the homotopy quotient by $\mathrm{GL}_n(\mathbb{F})$. Applying this to $\underline{*}_{>0}$, we get

Definition 2.2. $\mathbf{R} \coloneqq r_* \underline{*}_{>0} \in \mathsf{Alg}_{E_\infty}(\mathsf{sSet}^{\mathbb{N}})$.

Since $\underline{*}_{>0}$ is contractible in positive ranks, the underlying object of the E_∞-algebra $\mathbf{R}$ is given by
$$n \mapsto \mathbf{R}(n) = * /\!\!/ \mathrm{GL}_n(\mathbb{F}) \simeq B\mathrm{GL}_n(\mathbb{F}) \qquad \text{for } n > 0.$$
If we ignore the additional *rank* n, this is indeed $\bigsqcup_{n\geq 1} B\mathrm{GL}_n(\mathbb{F})$.

As we are only interested in homology, we might as well take $\Bbbk$-valued singular chains:

Definition 2.3. $\mathbf{R}_\Bbbk \coloneqq (n \mapsto \Bbbk[\mathbf{R}(n)]) \in \mathsf{Alg}_{E_\infty}(\mathsf{sMod}_\Bbbk)$.

We did not yet define E_∞-algebras precisely. One way to do so is to define them as algebras over the $\mathcal{C}_\infty$-operad. This is the colimit as $k \to \infty$ of the operads $\mathcal{C}_k$ of *little k-cubes*, the map $\mathcal{C}_k \to \mathcal{C}_{k+1}$ given by $- \times [0,1]$. These encode increasingly commutative algebraic structure; E_1 being associative and E_∞ commutative (as usual, in a homotopical sense).

By restriction, $\mathbf{R}_\Bbbk$ can be considered as an E_k-algebra for every k. Our proof of the Nesterenko–Suslin theorem will depend on a "presentation" $\mathbf{R}_\Bbbk$ as an E_∞-algebra, but to understand this, we will also need to consider as an E_k-algebra. The case $k = 2$ will in particular play a distinguished role.

2.2. **Presenting an E_k-algebra.** For us, presenting $\mathbf{R}_\Bbbk$ as an E_k-algebra will mean to exhibit a weakly equivalent CW-E_k-algebra. Let me first describe the simpler notion of a cellular E_k-algebra. This is the straight-forward extension of the notion of a cellular space to the setting of E_∞-algebras in sMod_k.

[1]During the talk, it was suggested that the notation $r_!$ may be more appropriate.

A *cell attachment* in the category Top of topological spaces is a pushout

$$\begin{tikzcd} \partial D^d \arrow[r] \arrow[d] & X \arrow[d] \\ D^d \arrow[r] & X'. \end{tikzcd}$$

A *cellular space* is a sequential colimit of such cell attachments, starting with $\varnothing$ (otherwise it would be a relative cellular space).

To generalize the input for such a cell attachment, we first replace D^d with $D^{n,d}$, the d-cell concentrated in rank n, and similarly ∂D^d with $\partial D^{n,d}$. Suppose we are given a diagram

$$\begin{tikzcd} \partial D^{n,d} \arrow[r] \arrow[d] & U^{\mathbf{E}_k}\mathbf{A} \\ D^{n,d} & \end{tikzcd}$$

with $U^{\mathbf{E}_k} \colon \mathsf{Alg}_{E_k}(\mathsf{sMod}_{\Bbbk}^{\mathbb{N}}) \to \mathsf{sMod}_{\Bbbk}^{\mathbb{N}}$ the forgetful functor. Using the left adjoint $\mathbf{E}_k$ of $U^{\mathbf{E}_k}$, the free E_k-algebra functor, we get a diagram

$$\begin{tikzcd} \mathbf{E}_k(\partial D^{n,d}) \arrow[r] \arrow[d] & \mathbf{A} \\ \mathbf{E}_k(D^{n,d}). & \end{tikzcd}$$

Its pushout in $\mathsf{Alg}_{E_k}(\mathsf{sMod}_{\Bbbk}^{\mathbb{N}})$ is a *E_k-cell attachment.*

A *cellular E_k-algebra* is one which a sequential colimit of iterated pushouts

$$\begin{tikzcd} \mathbf{E}_k(\partial D^{n,d}) \arrow[r] \arrow[d] & \mathbf{A} \arrow[d] \\ \mathbf{E}_\infty(D^{n,d}) \arrow[r] & \mathbf{A}' \end{tikzcd}$$

starting with 0.

Intuitively, a CW-E_k-algebra is one where the attachments happen in order of dimension. More precisely, it is an additional structure given by a lift to an E_k-algebra in filtered objects, where d-dimensional cells are attached in the dth filtration step. This identifies what is most important about a CW-complex: its *skeletal filtration.* A CW-E_k-algebra by definition has such a filtration, whose associated graded is a free E_k-algebra with generators $S^{n,d}$ in filtration d.

2.3. **E_k-indecomposables.** The question is now: *which E_k-cells do we need to present $\mathbf{R}_k$?* Like for simply-connected spaces, or more relevant to us, non-negatively graded chain complexes, the answer is controlled by a homology theory. To define it, we use that there is a functor

$$Q^{E_k} \colon \mathsf{Alg}_{E_k}(\mathsf{sMod}_k^{\mathbb{N}}) \longrightarrow \mathsf{sMod}_k^{\mathbb{N}}$$

of *E_k-indecomposables*, with the following properties:

- It is a left adjoint (so commutes with pushouts and sequential colimits).
- It satisfies $Q^{E_k}(\mathbf{E}_k(X)) \simeq X$, naturally in X.

We then define E_k-homology as:

$$H^{E_\infty}_{n,d}(\mathbf{A}) \coloneqq H_d(\mathbf{A}(n)).$$

Remark 2.4. We do not need the definition of $Q^{E_k}(-)$, but seeing it defined may help your intuition. As the name E_k-indecomposables suggests, it is the (derived functor of) the construction which collapses to zero all elements that are in the image of a k-ary operations for $k \geq 2$. That is, it is defined by the exact sequence

$$\bigsqcup_{n \geq 2} \mathcal{C}_k(n) \otimes_{\Sigma_n} \mathbf{A}^{\otimes n} \longrightarrow \mathbf{A} \longrightarrow Q^{E_k}(\mathbf{R}) \longrightarrow 0.$$

Remark 2.5. Strictly speaking this is relative indecomposables for the map of operads $\mathcal{C}_k(1) \to \mathcal{C}_k$, but $\mathcal{C}_k(1)$ is contractible so it is the same as absolute indecomposables.

The two properties of Q^{E_k} imply that writing $\mathbf{A}$ as a cellular E_k-algebra gives a way to write $Q^{E_k}(\mathbf{A})$ as a cellular object in $\mathsf{sMod}_{\Bbbk}^{\mathbb{N}}$. In particular, the E_k-homology gives a lower bound on the number of E_k-cells needed. In fact, under a mild condition exactly that number of cells suffices; one needs the relative Hurewicz theorem which holds when $\mathbf{A}(0) \simeq 0$.

One does not usually compute E_k-homology directly from the definitions. Computations either use the two defining properties, or that it can be expressed in terms of iterated bar constructions. Variations of the following results where obtained by Getzler–Jones [GJ94], Basterra–Mandell [BM05], Francis [Fra13], and Fresse [Fre11], but we also gave a proof in [GKRW18a]:

Theorem 2.6. *If $\mathbf{A}^+ \to \Bbbk$ denotes the unitalisation of $\mathbf{A}$, considered as an augmented unital E_∞-algebra, then we have*

$$\Bbbk \oplus \Sigma^k Q^{E_k}(\mathbf{A}) \simeq B^{E_k}(\mathbf{A}^+),$$

where B^{E_k} is a generalization of the k-fold iterated bar construction to E_k-algebras.

This description has two advantage:

- Bar constructions can be computed iteratively, i.e. $B^{E_\ell}(B^{E_k}(\mathbf{A})) \simeq B^{E_{k+\ell}}(\mathbf{A}^+)$ for an $E_{k+\ell}$-algebra $\mathbf{A}$. In particular, we can take $\ell = 1$ and get a bar spectral sequence

  $$\mathrm{Tor}^{\Bbbk \oplus H^{E_k}_{*,*}(\mathbf{A})}(\Bbbk, \Bbbk) \Longrightarrow H^{E_{k+1}}_{*,*}(\mathbf{A}^+).$$

  This means vanishing lines for E_1- or E_2-homology of $\mathbf{R}_\Bbbk$ can be transferred to E_∞-homology.
- Bar construction of $r_* \underline{*}_{>0}$ can be computed in terms of certain complexes. As bar constructions are colimits, and r_* is a left adjoint, we can compute it "upstairs" in terms of the unitalization $\underline{*}$ of $\underline{*}_{>0}$ and the Day convolution monoidal structure. The answer for $\mathbf{R}_\Bbbk$ will be in terms of the split E_k-building, as we will discuss now.

3. The split E_2-buildings

Let $\mathsf{Sub}(M)$ be the poset of summands of M ordered by inclusion.

Definition 3.1. A *splitting* indexed by a pointed set X is a function $f\colon X \to \mathsf{Sub}(M)$ such that the induced map $\bigoplus_{x \in X} f(x) \to M$ is an isomorphism.

Definition 3.2. The *split k-fold building* $\widetilde{D}^k(M)_{\bullet,\ldots,\bullet}$ is obtained from the functor

$$\begin{aligned} \mathsf{FinSet}_* &\longrightarrow \mathsf{Set}_* \\ X &\longmapsto \frac{\text{splittings } f\colon X \to \mathsf{Sub}(M) \text{ into modules indexed by elements of } X}{\text{splittings with } f(*) \neq 0}. \end{aligned}$$

by precomposition with $(\Delta^{\mathrm{op}})^k \to \mathsf{FinSet}_*$ sending $(p_1, \ldots, p_k)$ to $[p_1, \ldots, p_k]_+$.

This natural in isomorphisms of M, so comes with a $\mathrm{GL}(M)$-action.

Example 3.3. If $k = 1$ and A is a Dedekind domain, this is a double suspension of Charney's split complex [Cha80].

Specializing to $A = \mathbb{F}$ a field, we get the promised description of E_k-homology of $\mathbf{R}$ in terms of some complex:

Proposition 3.4. $H^{E_k}_{n,d}(\mathbf{R}) \cong H_{d-k}(\widetilde{D}^k(\mathbb{F}^n) /\!\!/ \mathrm{GL}_n(\mathbb{F}))$.

To say something about these complexes, we compare them to the more familiar non-split building. Let $[p]$ be the poset $0 < 1 < \cdots < p$, and $[p_1, \ldots, p_k] \coloneqq [p_1] \times \cdots \times [p_k]$. Furthermore, let $[a_1 \leq b_1] \times \cdots \times [a_k \leq b_k]$ be the k-cube which corners given by $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ and $[a_1 \leq b_1] \times \cdots \times [a_k \leq b_k] \setminus b$ the punctured cube obtained from it by deleting b.

Definition 3.5. A *lattice* in an A-module M is a functor $\varphi\colon [p_1, \ldots, p_k] \to$ summands of M such that

$$\operatorname*{colim}_{[a_1 \leq b_1] \times \cdots \times [a_k \leq b_k] \setminus b} \varphi \longrightarrow \varphi(b)$$

is the inclusion of a summand. It is *full* if $\varphi(a_1, \ldots, a_k) = 0$ if $a_i = 0$ for some i, and $\varphi(p_1, \ldots, p_k) = M$.

This is a k-fold simplicial object, as is the sub-object of non-full lattices.

Definition 3.6. The *non-split k-fold building* $D^k(M)_{\bullet,\ldots,\bullet}$ is the pointed k-fold simplicial set with $(p_1, \ldots, p_k)$-simplices given by the pushout

$$\begin{tikzcd} \text{non-full lattices} \arrow[r] \arrow[d] & \text{lattices} \arrow[d] \\ * \arrow[r] & D^k(M)_{p_1,\ldots,p_k} \end{tikzcd}$$

Example 3.7. For $k = 1$ and $A = \mathbb{F}$ this is a familiar object: it is the double suspension of the *Tits building* $\mathcal{T}(\mathbb{F}^n)$, the poset of non-trivial proper subspaces ordered by inclusion. The double suspension appears because we allow the elements 0 and $\mathbb{F}^n$.

This comes with an action of $\mathrm{GL}(M)$. Using the maps $\Delta^1/\partial\Delta^1 \wedge D^k(M)_{\bullet,\ldots,\bullet} \to D^{k+1}(M)_{\bullet,\ldots,\bullet}$ given on the unique non-degenerate simplex $0 < 1$ of Δ^1, these assemble to a spectrum $\mathbf{D}(M)$, *Rognes' stable building* [Rog92], with $\mathrm{GL}(M)$-action. We will revisit this in Remark 3.14.

Forgetting the splittings gives a $\mathrm{GL}(M)$-equivariant map $\widetilde{D}^k(M) \to D^k(M)$, and thus a map on homotopy quotients

$$\widetilde{D}^k(M) /\!\!/ \mathrm{GL}(M) \longrightarrow D^k(M) /\!\!/ \mathrm{GL}(M).$$

The lattice condition is so that the map $\widetilde{D}^k(M)_{p_1,\ldots,p_k} \to D^k(M)_{p_1,\ldots,p_k}$ is surjective; a "k-dimensional filtration" satisfies the lattice condition if and only if it comes from a splitting. Thus this map induces an isomorphism on homology if and only if the map from the stabilizer of a splitting to the stabilizer of its underlying lattice is always a homology isomorphism.

The most basic case, for $k = 1$ and a single step, is the inclusion of block-diagonal matrices into the block upper-diagonal matrices. If this induces an isomorphism of homology we say A satisfies the *Nesterenko–Suslin property*. This basic case implies the general case by induction. There is a result of Nesterenko and Suslin that it induces an isomorphism on homology when A has *many units* [NS89].

Example 3.8. This is satisfied is there are elements $a_1, a_2, \ldots$ such $\sum_{I \subset \mathbb{N} \text{ finite}} a_i$ is always a unit. This is true for infinite fields.

Thus we conclude that:

Proposition 3.9. *For an infinite field $\mathbb{F}$, the map*

$$\widetilde{D}^k(\mathbb{F}^n) /\!\!/ \mathrm{GL}_n(\mathbb{F}) \longrightarrow D^k(\mathbb{F}^n) /\!\!/ \mathrm{GL}_n(\mathbb{F}).$$

Example 3.10. The Solomon–Tits theorem says the Tits building $\mathcal{T}(\mathbb{F}^n)$ is $(n-3)$-connected [Sol69], so $D^k(\mathbb{F}^n)$ is $(n-1)$-connected. Thus $H^{E_1}_{n,d}(\mathbf{R}_k) = 0$ for $d < n-1$ (remember there was a shift by k).

Furthermore $\widetilde{H}_{n-2}(\mathcal{T}(\mathbb{F}^n))$ is a $\mathbb{Z}[\mathrm{GL}_n(\mathbb{F})]$-module known as the Steinberg module $\mathrm{St}(\mathbb{F}^n)$. This admits an explicit presentation in terms of *apartments*: it is generated by symbols $[v_1, \ldots, v_n]$ for $(v_1, \ldots, v_n)$ an ordered basis of $\mathbb{F}^n$, satisfying the relations:

(1) $[av_1, v_2, \ldots, v_n] = [v_1, \ldots, v_n]$ for $a \in \mathbb{F}^\times$,
(2) $[v_{\sigma(1)}, \ldots, v_{\sigma(n)}] = \mathrm{sign}(\sigma)[v_1, \ldots, v_n]$ for $\sigma \in \Sigma_n$,
(3) $[v_1, v_2, v_3, \ldots, v_n] = [v_1 + v_2, v_2, v_3, \ldots, v_n] + [v_1, v_1 + v_2, v_3, \ldots, v_n]$.

We get $H^{E_1}_{n,n-1}(\mathbf{R}_\mathbb{Z}) = (\mathrm{St}(\mathbb{F}^n))_{\mathrm{GL}_n(\mathbb{F})}$, the $\mathrm{GL}_n(\mathbb{F})$-coinvariants of the Steinberg module. These coinvariants are easily shown to vanish for $n \geq 2$: as $\mathrm{GL}_n(\mathbb{F})$ acts transitively on bases the coinvariants are generated by $[e_1, \ldots, e_n]$ and then (3) can be written as $[e_1, \ldots, e_n] \sim 2[e_1, \ldots, e_n]$. Thus we get another degree of vanishing of E_1-homology.

However, we can do better. Any k-dimensional lattice gives us a k-tuple of 1-dimensional lattices by restricting to the lines $(p_1, \ldots, p_{i-1}, j, p_{i+1}, \ldots, p_k)$. That is, there is an injective map

$$D^k(M)_{p_1,\ldots,p_k} \to D^1(M)_{p_1} \wedge \cdots \wedge D^1(M)_{p_k}.$$

For $k = 2$ (and also $k = 1$) this is an isomorphism! That is, we can recover the 2-dimensional lattice from the horizontal and vertical filtrations. This is because

$$\begin{tikzcd} V_1 \arrow[r] & V_{12} \\ V_0 \arrow[u] \arrow[r] & V_2 \arrow[u] \end{tikzcd}$$

is a lattice if and only if $V_0 = V_1 \cap V_2$.

Theorem 3.11 (GKRW)**.** *The map $D^2(\mathbb{F}^n) \to D^1(\mathbb{F}^n) \wedge D^1(\mathbb{F}^n)$ is a weak equivalence, and thus the left hand side is $(2n-2)$-connected. Thus we get that*

$$H^{E_2}_{n,d}(\mathbf{R}_\mathbb{Z}) = \begin{cases} 0 & \text{if } d < 2n-2, \\ (\mathrm{St}(\mathbb{F}^n) \otimes \mathrm{St}(\mathbb{F}^n))_{\mathrm{GL}_n(\mathbb{F})} & \text{if } d = 2n-2. \end{cases}$$

This leads to the question what these *double Steinberg coinvariants* are? As $\mathcal{T}(\mathbb{F}^n)$ is not just $(n-3)$-connected but $(n-2)$-dimensional, $\mathrm{St}(\mathbb{F}^n)$ is a subspace of the cellular chains $\widetilde{C}_{n-2}(\mathcal{T}(\mathbb{F}^n))$, the free abelian group on the full flags in $\mathbb{F}^n$. We can define a non-degenerate bilinear form

$$\widetilde{C}_{n-2}(\mathcal{T}(\mathbb{F}^n)) \otimes \widetilde{C}_{n-2}(\mathcal{T}(\mathbb{F}^n)) \longrightarrow \mathbb{Z}$$

by declaring the full flags are an orthonormal basis. This is evidently $\mathrm{GL}_n(\mathbb{F})$-invariant.

Theorem 3.12 (GKRW)**.** *For $n \geq 1$, the restriction of this bilinear form to $\mathrm{St}_n(\mathbb{F}^n)$ gives map*

$$(\mathrm{St}(\mathbb{F}^n) \otimes \mathrm{St}(\mathbb{F}^n))_{\mathrm{GL}_n(\mathbb{F})} \longrightarrow \mathbb{Z}$$

which is an isomorphism.

In fact, there is a ring structure on $\bigoplus_{n\geq 1} H^{E_2}_{n,2(n-1)}(\mathbf{R}_\mathbb{Z})$ (this comes from an E_∞-structure on $B^{E_2}(\mathbf{R}^+_\mathbb{Z})$), and these $\mathbb{Z}$'s assemble to a divided power algebra on a single generator. We can now iterate the bar spectral sequence to get a bounds on the E_∞-homology of $\mathbf{R}_\mathbb{Z}$.

Theorem 3.13 (GKRW)**.** *We have that $H^{E_\infty}_{n,d}(\mathbf{R}_\mathbb{Z}) = 0$ for $d < 2n-2$. For $d = 2n-2$, we get the indecomposables in the divided power on a single generator, which are torsion for $n \geq 2$.*

Remark 3.14. This proves a conjecture of Rognes [Rog92]. He defined a stable rank filtration on the algebraic K-theory spectrum $K(\mathbb{F})$ (his construction is more general) with filtration quotients

$$\mathbf{D}(\mathbb{F}^n) /\!\!/ \mathrm{GL}_n(\mathbb{F}),$$

the homotopy quotient of the stable buildings. He conjectured that $\mathbf{D}(\mathbb{F}^n)$ was $(2n-3)$-connected and the $\mathrm{GL}_n(\mathbb{F})$-coinvariants of $\widetilde{H}_{2n-2}(\mathbf{D}(\mathbb{F}^n))$ are torsion. Combining our vanishing result with the comparison between non-split and split building, we prove a version of this conjecture after taking homotopy quotients; this seems to suffice for all intended applications.

Let me elaborate on this filtration of $\mathbf{K}(\mathbb{F})$; taking rational spectrum homology, with concides with rational homotopy, we get a spectral sequence converging to $K_i(\mathbb{F}) \otimes \mathbb{Q}$ with a vanishing line of slope 2 on its E^2-page; that is, the induced filration on $K_i(\mathbb{F}) \otimes \mathbb{Q}$ has about $i/2$ non-zero steps. From the computations I will tell you about now, and Rognes' computations [Rog10], this seems like an interesting filtration.

4. A presentation of $\bigsqcup_n B\mathrm{GL}_n(\mathbb{F})$

In Theorem 3.13, we obtained a vanishing line for the E_∞-homology of $\mathbf{R}_\Bbbk$ of slope 2, where we recall that $H_{n,d}(\mathbf{R}_\Bbbk(n)) = H_d(B\mathrm{GL}_n(\mathbb{F}); \Bbbk)$. It thus remains to understand which E_∞-cells are needed for low n and d. This can be done by explicitly determining the homology of the groups $\mathrm{GL}_1(\mathbb{F})$ and $\mathrm{GL}_2(\mathbb{F})$ in a range and invoking the Hurewicz theorem. For the sake of simplicitly, I will take $\Bbbk = \mathbb{Q}$. The result is Figure 1.

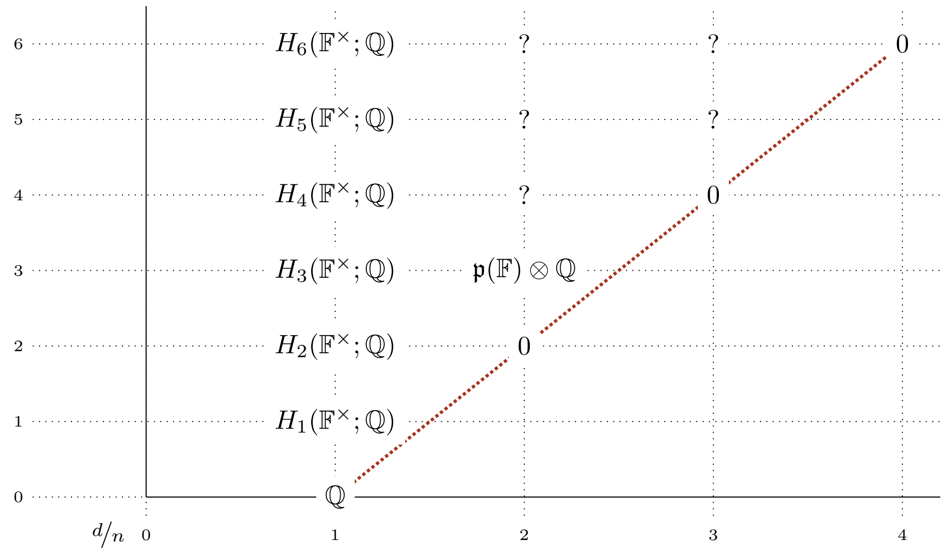

FIGURE 1. The E_∞-homology of $\mathbb{R}_\mathbb{Q}$.

- The first case is easy; $\mathrm{GL}_1(\mathbb{F}) = \mathbb{F}^\times$. This suggests to take the free E_∞-algebra on $B\mathbb{F}^\times$ in rank 1 as a first approximation:

$$\mathbf{A}_\mathbb{Q} \longrightarrow \mathbf{R}_\mathbb{Q}.$$

  This E_∞-algebra is given explicitly by $\mathbf{A}(n) = B(\Sigma_n \ltimes (\mathbb{F}^\times)^n) = B\mathrm{GM}_n(\mathbb{F})$, the *monomial matrices.*
- In Suslin's work on $K_3(\mathbb{F})$, he computed that

$$H_*(B\mathrm{GL}_2(\mathbb{F}), B\mathrm{GM}_2(\mathbb{F}); \mathbb{Q}) = \begin{cases} 0 & \text{if } * = 1, 2, \\ \mathfrak{p}(\mathbb{F}) \otimes \mathbb{Q} & \text{if } * = 3, \end{cases}$$

  the latter being the *pre-Bloch group* (it admits a nice description in terms of generators and relations, but there is no reason to include that know). It's generated by $\{a\}$ for $a \in \mathbb{F}^\times \setminus \{1\}$ and has the relations $\{x\} - \{y\} + \{\frac{y}{x}\} + \{\frac{1-x^{-1}}{1-y^{-1}}\} + \{\frac{1+x}{1+y}\}$ whenever $x, y, 1-x, 1-y, x-y \in \mathbb{F}^\times \setminus \{1\}$. This is enough information to compile the diagram for the E_∞-homology of $\mathbf{R}_\mathbb{Q}$.

We want to understand the relative homology groups

$$H_d(\mathrm{GL}_n(\mathbb{F}), \mathrm{GL}_{n-1}(\mathbb{F}); \mathbb{Q}).$$

Working over the rationals, this can be done by removing the cell in bidegree $(n, d) = (1, 0)$ (over the integers, more care is required as there is a difference between killing σ in underlying objects and in E_∞-algebras.) This deletes $\mathbb{Q}$ from the entry $(n, d) = (1, 0)$ in Figure 1.

The CW approximation results for E_∞-algebras gives us a CW-E_∞-algebra $\mathbf{B}$ weakly equivalent to $\mathbf{R}'_\mathbb{Q} := \mathbf{R}_\mathbb{Q} \cup_\sigma D^{1,1}$; this has E_∞-cells only on or above the line $d = n$. Now, the associated graded of the filtration coming with the CW-sequence is given by the free E_∞-algebra on the E_∞-cells; rationally this is just the free graded-commutative algebra.

Thus we get a spectral sequence

$$\Lambda^*(x_{n_i,d_i}) \Longrightarrow H_{n,d}(\mathbf{R}'_{\mathbb{Q}}),$$

and the domain vanishes below the line $d = n$. Thus we read off the first part of the Nesterenko–Suslin result. We also see that on this line it is generated by $\mathbb{F}^\times \otimes \mathbb{Q}$. The d^1-differential is given in terms of the attaching map and $\mathfrak{p}(\mathbb{F})$ is attached along a map imposing the relation $\{a\}\{b\} = 0$ if $a + b = 1$ (so as to make $H_2(B\mathrm{GL}_2(\mathbb{F}); \mathbb{Q})$ correct). This tells us that the second part of the Nesterenko–Suslin theorem.

This type of reasoning makes one think that the next line

$$\bigoplus_{n\geq 0} H_{n,n+1}(B\mathrm{GL}_n(\mathbb{F}), B\mathrm{GL}_{n-1}(\mathbb{F}); \mathbb{Q}) = \bigoplus_{n\geq 0} H_{n,n+1}(\mathbf{R}'_{\mathbb{Q}})$$

might in some sense be generated by Milnor K-theory and the cells on that line. It is a $K^M_*(\mathbb{F}) \otimes \mathbb{Q}$-module, and the precise statement about its generators is as follows:

Theorem 4.1 (GKRW)**.** *There is a map*

$$\mathrm{Harr}_3(K^M_*(\mathbb{F}) \otimes \mathbb{Q})_n \longrightarrow \mathbb{Q} \otimes_{K^M_*(\mathbb{F})\otimes\mathbb{Q}} \bigoplus_{n\geq 0} H_{n,n+1}(\mathbf{R}'_{\mathbb{Q}})$$

which is an isomorphism for $n \geq 4$.

In particular, if $K^M_*(\mathbb{F}) \otimes \mathbb{Q}$ is Koszul then the third Harrison homology is supported in degree 3 and this is a finite generation result. It follows from conjectures of Parshin and Beilinson [Pos11], which are still open, that rationalized Milnor K-theory *is* Koszul when $\mathbb{F}$ is of finite characteristic.

Remark 4.2. Why does Harrison homology show up? Firstly, Harrison homology is the name for rational E_∞-homology. Now, the right hand side is computed by the lowest non-zero homology groups of the bar construction $B(\mathbb{Q}, \mathbf{R}'_{\mathbb{Q}}, \mathbf{I})$ with $\mathbf{I}$ defined by the fiber sequence of $\mathbf{R}'_{\mathbb{Q}}$-modules

$$\mathbf{I} \longrightarrow \mathbf{R}'_{\mathbb{Q}} \longrightarrow \mathbf{K}_{\mathbb{Q}},$$

with $H_{*,*}(\mathbf{K}_{\mathbb{Q}})$ equal to $K^M_*(\mathbb{Q})$ on the line $d = n$ and 0 otherwise, obtained by killing higher degrees elements. The homology of this bar construction can expressed in terms of the relative E_∞-homology of $\mathbf{R}'_{\mathbb{Q}}$ and $\mathbf{K}$. By the long exact sequence for E_∞-homology, this is related to the E_∞-homology of both $\mathbf{R}'_{\mathbb{Q}}$ and $\mathbf{K}$. We know that for $\mathbf{R}'_{\mathbb{Q}}$ this has a vanishing line of slope 2 in its E_∞-homology, but for $\mathbf{K}$ this amounts to the vanishing of certain Harrison homology groups.

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