

Locally compact abelian groups

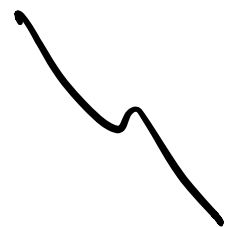


• Reminders

κ = fixed cardinal.

$$\text{Cond}_\kappa(\text{Sets}) := \text{Shv}_{\text{epi}}(\text{Pro Fin}_\kappa).$$

$$\text{Proj} \hookrightarrow \text{Pro Fin}_\kappa \hookrightarrow \text{CHaus}_\kappa.$$



Extremally disconnected sets.

$\equiv$

► analogy

$$P_{\Sigma}(\text{Proj}) \xrightarrow{\sim} \text{Shv}_{\text{epi}}(\text{Pro Fin}_k) \xrightarrow{\sim} \text{Shv}_{\text{epi}}(\text{CHaus}_k).$$

$\parallel$
 $\text{Cond}_k(\text{Sets})$

$\swarrow$
 1) $\emptyset \mapsto *$

2) $W \mapsto x$

Upshoe

1)

2).

$$\text{Cond}_k(-) : \text{Cat}_{\infty}^{\text{fp}} \longrightarrow \widehat{\text{Cat}_{\infty}^{\text{fp}}}$$

$$\blacktriangleright \mathcal{C}_\ell \longmapsto$$

$$\blacktriangleright \text{Ab}(\text{Cond}_k) \simeq \mathcal{P}_{\Sigma}(\text{Proj} \text{ Ab})$$

Thm $\text{Ab}(\text{Cond}_K)$ is a Grothendieck
abelian category.

$\{P : P \in \text{Proj}\}$ forms compact

projective generators

- $\mathbb{Z}[-]: \text{Cond}_K \rightleftarrows \text{Ab}(\text{Cond}_K)$

$$\mathbb{Z}[X] = L_{\text{epi}}(T \mapsto \mathbb{Z}[X(T)]).$$

- $\underline{\text{Hom}}(M, N)(T)$

sheaf Hom

$$\triangleright D(\text{Cond}_k(\text{Ab})) \leftarrow \text{Compact generators.} =$$

$$\triangleright \text{has } R\text{Hom}(M, N)$$

$$\text{Maps}(T, R\text{Hom}(M, N)) \simeq \text{Maps}(,)$$

$$\bullet D(\text{Cond}_k(\text{Ab})) \simeq \text{Cond}_k(D(\text{Ab}))$$

$$\bullet \text{Fun}_\Sigma(\text{Proj}_k^{\text{op}}, D(\text{Ab})) \simeq \text{Fun}_\Sigma(\text{Proj}_k^{\text{op}} \times (,), \text{Spec})$$

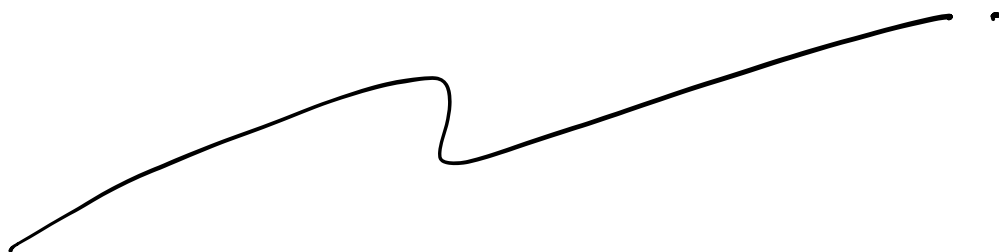
- $D(\text{Cond}_k(\text{Ab})) \simeq \text{Cond}_k(D(\text{Ab}))$

- $\text{Fun}_{\Sigma}(\text{Proj}_k^{\text{op}}, D(\text{Ab}))$

- $\simeq \text{Fun}_{\Sigma}(\text{Proj}_k^{\text{op}} \times - , \mathcal{G})$

is :

$$D(\text{Cond}(\text{Ab}))_{\geq 0}$$



$$\text{Cond}(\text{Set}) \ni X$$

$$\triangleright H_{\text{sheaf}}^i(X; \mathbb{Z})$$

$$\triangleright H_{\text{Cond}}^i(X; \mathbb{Z})$$

$$S_0 \rightarrow X \rightsquigarrow 0 \rightarrow \underset{\text{Cond}(\text{Ab})}{\text{Hom}(\mathbb{Z}[S_0], \mathbb{Z})} \rightarrow \dots$$

$$\sim$$

$$S_i = \text{extr. disconnected sets. } (\mathbb{P}(Y))$$

Thm $X = \text{cpct Haus}$:

$$H_{\text{sheaf}}^i(X; \mathbb{Z}) \cong H_{\text{cond}}^i(X; \mathbb{Z}) \\ = \text{Ext}$$

Thm $X \text{ cpct Haus}$:

$$H_{\text{cond}}^i(S; \mathbb{R}) = 0 = \text{Ext.}$$

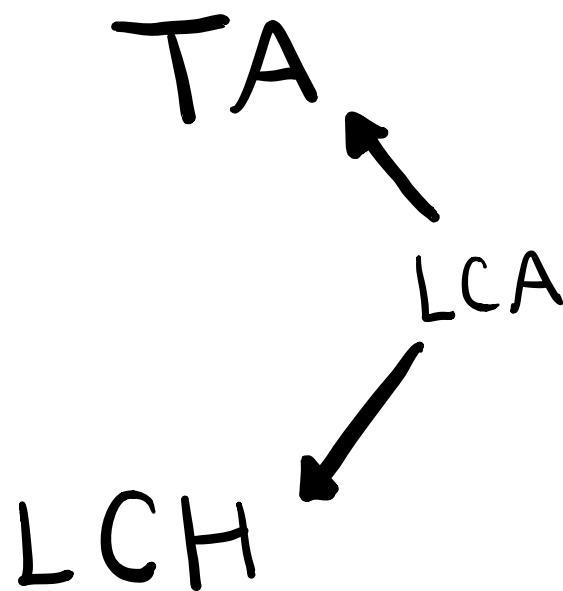
$$(\Leftrightarrow) S_{\bullet} \longrightarrow S$$

$$0 \rightarrow C(S; \mathbb{R}) \rightarrow C(S_{\bullet}; \mathbb{R}) \rightarrow \dots C(S_j; \mathbb{R})$$

is exact.

Goal:

- expand our library of computations



- $D^b(LCA)$

- $A' \xrightarrow{\text{closed Sub}} A \rightarrow A/A'$

is a LCA

- $\text{cok}(B \xrightarrow{f} A) := A / \overline{f(B)}$

► $\mathbb{Q} \hookrightarrow \mathbb{R}$ monic

► $\overline{\mathbb{Q}} = \mathbb{R}$ epi

► In an abelian cat,
mono + epi $\Rightarrow$ isomorphism

$$\underline{\text{Thm}} \quad \cdot \quad \underline{A} = \underline{\prod_I T} \in$$

i) $M = \text{discrete.}$

$$\blacktriangleright R\text{Hom}(\underline{R}, \underline{M}) = 0$$

$$\blacktriangleright R\text{Hom}(\underline{A}, \underline{M}) = \bigoplus_I \underline{M}[-i]$$

$$\{\text{type } R\} \subseteq {}^\perp \{\text{type } S\}$$

$$\{S\} \longrightarrow \{\text{type } S'\}[+1].$$

$$2) \quad R\text{Hom}(\underline{A}, \mathbb{R}) = 0.$$

$$\{\text{type } \mathbb{S}'\} \subseteq^\perp \{\text{type } \mathbb{R}\}$$

Classification of L.C.A.

- discrete. tors free $(\mathbb{Z}, \mathbb{Q})$
- discrete tors $(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}, \mathbb{Q}_p/\mathbb{Z}_p)$
- Compact Lie $(\mathbb{T})$
- noncompact Lie $(\mathbb{R})$

$$\underline{\text{Thm.}} \quad A = \text{LCA}$$

$$\bullet \quad A \cong A' \times \mathbb{R}^n$$

$$\bullet \quad \text{discrete} \rightarrow A' \rightarrow \text{Compact}$$

$$(\kappa \rightarrow D[+1] \in \text{Ext}^1(\kappa, D) \simeq \bigoplus_{\kappa(\kappa)} D)$$

$$\bullet \quad \text{LCA} \longrightarrow \text{LCA}^{\text{op}}$$

$$A \longmapsto \text{ID}(A)$$

$$\{\text{Compact}\} \longleftrightarrow \{\text{discrete}\}.$$

Bonus:

Thm (Clauven.)

$$\bigoplus_{p \in \text{Spec } \mathbb{Z}} K(\mathbb{F}_p)$$



$$K(\mathbb{Z}) \rightarrow K(\mathbb{R}) \rightarrow K(\text{LCA})$$



$$K(\mathbb{Q})$$

Thm (Brauer)

$$F \supseteq \mathcal{O}$$

$$LCA_{\mathcal{O}}$$

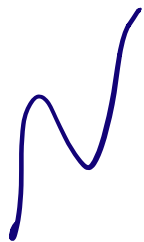
$$K(\mathcal{O}) \rightarrow K(\mathbb{R})^{\times} \oplus K(\mathbb{C})^{\times}$$

$$\rightarrow K(LCA_{\mathcal{O}})$$

Lemma A, B Hausdorff TA

Say A is compactly generated.

$$\underline{\text{Hom}}(A, \underline{B}) \cong \underline{\text{Hom}}(A, B)$$



Pf)

$$\underline{\text{Hom}(A, B)(T)}$$

$$= \text{Hom}_{\text{cts}}(T, \text{Hom}_{\text{co}}(A, B))$$

$$\simeq \text{Hom}_{\text{cts}}(T \times A, B)$$

$$\simeq \text{Hom}_{\text{Cond}}(\underline{A} \times T, B)$$

$$\simeq \text{Hom}(\mathbb{Z}[\underline{A} \times T], B)$$

$$\mathbb{Z}[\underline{A} \times T] = \mathbb{Z}[\underline{A}] \otimes \mathbb{Z}[T]$$



$$\underline{A} \otimes \mathbb{Z}[T]$$



$$\mathbb{Z}[\underline{A}] \longrightarrow \underline{A}$$

epi

(I_{inj} .)

$$\underline{\text{Hom}}(A, B)(T) = \text{Hom}(A \otimes \mathbb{Z}[T], B)$$



$$\underline{\text{Hom}}(A, B)(T) = \text{Hom}(\mathbb{Z}[A \times T], B)$$

$$\bullet \text{Hom}_{\text{Ab}(\text{Grp})}(\mathbb{Z}[A \times T], B)$$

$$\simeq \text{Hom}_{\text{Cond}}(A \times T, B)$$

$$\simeq \text{Hom}_{\text{Cts}}(T \times A, B) \simeq \text{Hom}(T, \text{Hom}(A, B))$$

(Surj.)

$$\bullet \quad \mathbb{Z}[\underline{A} \times \underline{A}] \longrightarrow \mathbb{Z}[\underline{A}] \longrightarrow \underline{A} \longrightarrow 0$$


$$\text{Ker} = \{ [a+b] - [a] - [b] \}$$

$$\bullet \quad T \longrightarrow \text{Hom}(A, B)$$

$$\text{Want: } \mathbb{Z}[\underline{A}] \otimes \mathbb{Z}[T] \longrightarrow \underline{B}$$

$$\begin{array}{ccc} & \downarrow & \nearrow \\ \mathbb{Z}[\underline{A}] \otimes T & & ? \end{array}$$

$$\begin{array}{ccc} \Leftrightarrow & \mathbb{Z}[\underline{A} \times \underline{A}] \otimes \mathbb{Z}[T] & \xrightarrow{0} \\ & \downarrow & \searrow \\ & \mathbb{Z}[\underline{A}] \otimes \mathbb{Z}[T] & \longrightarrow \underline{B} \end{array}$$

$$\Leftrightarrow \begin{array}{ccc} A \times A \times T & \xrightarrow{0} & B \\ \downarrow & \nearrow \neq & \\ A & & \end{array}$$

$$\Leftrightarrow \begin{array}{ccc} A \times A \times t & \xrightarrow{0} & B \\ \downarrow & \nearrow \neq & \\ A & & \end{array}$$

□

Write $\underline{RHom}(A, B)$.

To proceed :

Thm (EM, Breen, Deligne.) $A \in \text{Ab}$

- $C_{\bullet}^{\text{can}}(A)$

with

- Functorial $(\text{Ab} \rightarrow K_{\geq 0}(\text{Ab}))$

- $C_n^{\text{can}}(A) = \bigoplus_{\text{finite}} \mathbb{Z}[A^{x \neq \text{finite}}].$

Canonical resolution

$$\bigoplus_{j=1}^{n_i} \mathbb{Z}[A^{r_{i,j}}]$$

$\vdots$



$$\mathbb{Z}[A^3] \oplus \mathbb{Z}[A^2]$$



$$\mathbb{Z}[A^2]$$



$$\mathbb{Z}[A]$$



$$A$$

Pf)

$$[\text{Step 1}] \quad A = \mathbb{Z}^{\oplus n} = \{p \in \text{Latt.}\}$$

If ok in this case:

$$C^{\text{con}}_*(P): \dots \rightarrow \mathbb{Z}[P^*] \rightarrow \mathbb{Z}[P] \rightarrow P$$

$$\begin{array}{ccc} \text{Latt} & \xrightarrow{\quad} & K(\text{Ab}) \\ \downarrow & & \nearrow \text{---} \\ P_{\Sigma}(\text{Latt}) & & \end{array}$$

$\simeq \text{Ab}$

[Step 2] WTS:

$C^{\text{con}}(P)$ exists for PG Latt.

$\text{Fun}(\text{Latt}, \text{Ab}) \leftarrow \text{ab cat.}$

X is n -pseudo coh $\Leftrightarrow$

- $\text{Ext}^i(X, -)$ compact

for $i=0, \dots, n-1$

- length n compact projectives

$$C_{\bullet} =$$

$$\underline{\text{Lemma}} \rightarrow P_i \rightarrow \dots \rightarrow P_0 \rightarrow X$$

(not necc. a res.)

$$P_i = \text{cyclic projectives}$$

$$H_i(C) \text{ is } n-i-1$$

$$\text{- pseudocoh for } i=0, \dots, n-1$$

$$\text{then } X \text{ is } n\text{-pseudo coh.}$$

[Step 3] - Induction step 1

$$\mathbb{Z}[P^2] \rightarrow \mathbb{Z}[P] \rightarrow P$$

$$\mathbb{Z}[B^n P]$$

$$M_j = \text{f.g. free}$$

$$H_i = \begin{cases} M_n \otimes_{\mathbb{Z}} P & i=2n \\ \vdots & \\ M_1 \otimes_{\mathbb{Z}} P & i=n+1 \\ P & i=n \\ 0 & i=n-1 \\ \vdots & \\ 0 & i=0 \end{cases}$$

$$\bullet \quad Z[B^n P] \rightarrow P$$

$$[-n]$$

$$\blacktriangleright H_i = n - i - 1 \quad \text{pseudo coh.} \quad i \in \{1, \dots, n-1\}$$

$$\blacktriangleright H_0 = n - 1 \quad \text{pseudo coh.}$$

$\Rightarrow$ done



Lemma

$$1) \quad \underline{R\text{Hom}}(\mathbb{R}, M) \simeq 0.$$

$$2) \quad \underline{R\text{Hom}}(\mathbb{T}, M) \simeq M[-1]$$

Pf)

$$0 \rightarrow \underline{\mathbb{Z}} \rightarrow \underline{\mathbb{R}} \rightarrow \underline{\mathbb{T}} \rightarrow 0.$$

So $1) \Rightarrow 2).$

$T \in \text{Cond.}$;

local to global S-S :

$$H_{\text{Cond}}^P(T, \text{Ext}^a(A, M))$$

$\Downarrow$

$$\underline{\text{Ext}}^{\text{ptg}}(A, M)(T)$$

But

$$C_{\bullet}^{\text{can}}(A) \otimes \mathbb{Z}[s]$$

}

$$R\text{Hom}(-, M)$$

}

$$H^2(A \times S_j M) \rightarrow \prod_{j=1}^{k_n} H^2(A^{\Gamma_{k_n, j}} \times S_j M)$$

$$H^1(A \times S_j M) \rightarrow \prod_{j=1}^{k_n} H^1(A^{\Gamma_{k_n, j}} \times S_j M)$$

...

$$H^0(A \times S_j M) \rightarrow \prod_{j=1}^n H^0(A^{\Gamma_{k_n, j}} \times S_j M)$$

||

⌊

$$\underline{\text{Ext}}^{p+q}(A, M)(S)$$

$$\cdot \quad 0 \longrightarrow \mathbb{R} \quad \quad \quad {}_{12}^0$$

$$\underline{\text{RHom}}(\mathbb{R}, \mathbb{R}) \longrightarrow \underline{\text{RHom}}(0, \mathbb{R})$$

· Claim boils down to

$$H_{\text{sheaf}}^i(S \times \mathbb{R}; M) \cong H_{\text{sheaf}}^i(S; M)$$



Lemma

$$\varinjlim_{J \subseteq I} \underline{R\mathrm{Hom}}(\prod_J \pi, M)$$

$|_2$

$$\underline{R\mathrm{Hom}}(\prod_I \pi, M).$$

Lemma

$$R\underline{\mathrm{Hom}}(A, \mathbb{R}) = 0$$

pf)

$$C^{\mathrm{con}}_*(A) \otimes \mathbb{Z}[T]$$

$\downarrow$

$$\underline{R\underline{\mathrm{Hom}}}(A, \mathbb{R})(T)$$

$\downarrow$

$$C(T; \mathbb{R})$$

$$= \text{cts}$$

$\mathbb{R}$ -valued fns

0



$$C(A \times T; \mathbb{R})$$



$$C(A^{x^2} \times T; \mathbb{R})$$



$$C(A^{x^2} \times T; \mathbb{R}) \oplus C(A^{x^3} \times T; \mathbb{R})$$



⋮

$$\bigoplus_{\text{finite}} C(A^{x_{\text{fin}}} \times T; \mathbb{R})$$



⋮

$$\underbrace{\qquad\qquad\qquad}_{C(A, T)}$$

Claim:

$$2_{\mathbb{R}}^i C^i(A, T) \xrightarrow{[2]_A} C^i(A, T)$$

are homotopic via h^i

Assuming claim:

$$f, df=0$$

$$\uparrow$$
$$C^i(A, T)$$

$$\blacktriangleright 2f = [2]^*(f) + d(h_{i-1}^*(f))$$

$\Downarrow$

$$f = \frac{1}{2} [2]^*(f) + d(h_{i-1}^*(f))$$

$$= \frac{1}{2^n} [2^n]^*(f) + d\left(\frac{1}{2} h_{i-1}^*(f)\right)$$

$$+ \frac{1}{4} h_{i-1}^*([2]^*(f)) + \dots$$

$$+ \frac{1}{2^n} h_{i-1}^*([2^{n-1}]^*(f))$$

► $[2^n]^* f$ has bdd norm,

$$\| [2^n]^* f \| \leq \| f \|$$

$\Rightarrow$

$$f = d \left(\frac{1}{2} h_{i-1}^*(f) \right)$$

$$+ \frac{1}{4} h_{i-1}^*([2]^*(f)) + \dots \quad).$$

To summarize

domain.

	$\mathbb{T}$	$\mathbb{R}$	$\mathbb{Z}$
$\mathbb{T}$	$\mathbb{Z}[-1]$ ✱	$\mathbb{T} \rightarrow M.$	$\mathbb{T}$
$\mathbb{R}$	0	$\mathbb{R}$	$\mathbb{R}$
$\mathbb{Z}$	$\mathbb{Z}[-1]$	0	$\mathbb{Z}$

↑
codomain.

Thm

$$\underline{D^b(\text{LCA})} \hookrightarrow D(\text{Cond Ab})$$

bounded derived category
of LCA's.

- agrees w/ Spitzweck-
Hofmann theory.

• Projective dimension 1

Comparison of valuation
rings of finite rank