Locally compact abelian groups

- Reminders

  κ = fixed cardinal.

  $$\mathrm{Cond}_\kappa(\mathrm{Sets}) := \mathrm{Shv}_{\mathrm{epi}}(\mathrm{ProFin}_\kappa).$$

  $$\mathrm{Proj} \hookrightarrow \mathrm{ProFin}_\kappa \hookrightarrow \mathrm{CHaus}_\kappa.$$

  extremally disconnected sets.

  $=$

analogy

$$P_\Sigma(\mathrm{Proj}) \xrightarrow{\sim} \mathrm{Shv}_{epi}(\mathrm{ProFin}_\kappa) \xrightarrow{\sim} \mathrm{Shv}_{epi}(\mathrm{CHaus}_\kappa).$$

$$\mathrm{Shv}_{epi}(\mathrm{ProFin}_\kappa) = \mathrm{Cond}_\kappa(\mathrm{Sets})$$

$\wr$

1) $\emptyset \longmapsto *$

2) $\sqcup \longmapsto \times$

Upshoe

1)

2)

$$\mathrm{Cond}_\kappa(-) : \mathrm{Cat}_\infty^{fp} \longrightarrow \widehat{\mathrm{Cat}_\infty^{fp}}.$$

- $\mathcal{C}_\ell \longmapsto$

- $\mathrm{Ab}(\mathrm{Cond}_\kappa) \simeq P_\Sigma(\mathrm{Proj}; \mathrm{Ab})$

Thm $Ab(Cond_\kappa)$ is a Grothendieck abelian category.

$\{ P : P \in Proj \}$ forms compact projective generators.

- $\mathbb{Z}[-] : \mathrm{Cond}_\kappa \rightleftarrows \mathrm{Ab}(\mathrm{Cond}_\kappa)$

  $\mathbb{Z}[X] = L_{epi}\left(T \longmapsto \mathbb{Z}[X(T)] \right.$.

- $\underline{\mathrm{Hom}}(M, N)(T)$

  sheaf Hom

- $D(\mathrm{Cond}_\kappa(Ab))$ $\longleftarrow$ compact generators. $=$

- has $R\underline{\mathrm{Hom}}(M,N)$

$$\mathrm{Maps}(T, R\underline{\mathrm{Hom}}(M,N)) \simeq \mathrm{Maps}(\quad , \quad)$$

- $D(\mathrm{Cond}_\kappa(Ab)) \simeq \mathrm{Cond}_\kappa(D(Ab))$

- $\mathrm{Fun}_\Sigma(\mathrm{Proj}_\kappa^{op}, D(Ab)) \simeq \mathrm{Fun}_\Sigma(\mathrm{Proj}_\kappa^{op} \times (\quad), \mathrm{Spc})$

$\shortparallel$

- $D(\mathrm{Cond}_\kappa(Ab)) \simeq \mathrm{Cond}_\kappa(D(Ab))$

- $\mathrm{Fun}_\Sigma(\mathrm{Proj}_\kappa^{op}, D(Ab))$

- $\simeq \mathrm{Fun}_\Sigma(\mathrm{Proj}_\kappa^{op} \times - \, , \quad)$

is :

$D(\mathrm{Cond}(\mathrm{Ab}))_{\geq 0}$

$\mathrm{Cond}(\mathrm{Set}) \ni X$

- $H^i_{\mathrm{Sheaf}}(X;\mathbb{Z})$

- $H^i_{\mathrm{Cond}}(X;\mathbb{Z})$

$$S_\bullet \longrightarrow X \rightsquigarrow 0 \longrightarrow \mathrm{Hom}_{\mathrm{Cond(Ab)}}(\mathbb{Z}[S_0],\mathbb{Z}) \longrightarrow \cdots$$

S_i = extr. disconnected sets. $\left(\beta(Y)\right)$

Thm X = cpct Haus:

$$H^i_{sheaf}(X;\mathbb{Z}) \cong H^i_{cond}(X;\mathbb{Z})$$

$$= \mathrm{Ext}$$

Thm X cpct Haus:

$$H^i_{cond}(S;\mathbb{R}) = 0 = \mathrm{Ext}.$$

$$\Leftrightarrow \quad S_\bullet \longrightarrow S$$

$$0 \to C(S;\mathbb{R}) \to C(S_0;\mathbb{R}) \to \cdots C(S_j;\mathbb{R})$$

is exact.

Goal:

- expand our library of computations

$$\begin{tikzcd} \mathrm{TA} & \\ & \mathrm{LCA} \arrow[ul] \arrow[dl] \\ \mathrm{LCH} & \end{tikzcd}$$

- $D^b(\mathrm{LCA})$

- $A' \hookrightarrow A \twoheadrightarrow A/A'$
  closed sub
  is a LCA
- $\operatorname{cok}(B \xrightarrow{f} A) := A/\overline{f(B)}$
- $\mathbb{Q} \hookrightarrow \mathbb{R}$ monic
- $\overline{\mathbb{Q}} = \mathbb{R}$ epi
- In an abelian cat,
  mono + epi $\not\Rightarrow$ isomorphism

Thm . $\underline{A} = \prod_I \mathbb{T}_\epsilon$.

1) M = discrete.

- $R\mathrm{Hom}(\underline{\mathbb{R}}, \underline{M}) = 0$
- $R\underline{\mathrm{Hom}}(\underline{A}, \underline{M}) = \bigoplus_I \underline{M}[-1]$

$\{\text{type } \mathbb{R}\} \subseteq^{\perp} \{\text{type } \mathcal{S}\}$.

$\{\ \mathcal{S}\ \} \longrightarrow \{\text{type } \mathcal{S}'\ \}[+1]$.

2) $R\underline{\mathrm{Hom}}(\underline{A}, \mathbb{R}) = 0$.

$\{\text{type } S^1\} \overset{\perp}{\subseteq} \{\text{type } \mathbb{R}\}$

Classification of L.C.A.

- discrete. tors free $(\mathbb{Z}, \mathbb{Q})$
- discrete tors $(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}, \mathbb{Q}_p/\mathbb{Z}_p)$
- compact Lie $(\mathbb{T})$
- noncompact Lie $(\mathbb{R})$

Thm. $A = LCA$

- $A \cong A' \times \mathbb{R}^n$

- $\text{discrete} \longrightarrow A' \longrightarrow \text{compact}$

$\left(K \to D[+1] \in \mathrm{Ext}^1(K, D) \simeq \bigoplus_{rk(K)} D \right).$

- $$\begin{aligned} LCA &\longrightarrow LCA^{op} \\ A &\longmapsto \mathbb{D}(A) \end{aligned}$$

$$\{\text{Compact}\} \longleftrightarrow \{\text{discrete}\}.$$

Bonus:

Thm (Clausen.)

$$\begin{tikzcd}
\bigoplus_{p \in \mathrm{Spec}\,\mathbb{Z}} K(\mathbb{F}_p) \arrow[d] & & \\
K(\mathbb{Z}) \arrow[r] \arrow[d] & K(\mathbb{R}) \arrow[r] & K(\mathrm{LCA}) \\
K(\mathbb{Q}) & &
\end{tikzcd}$$

<u>Thm</u> (Brauling)

$F \supseteq \mathcal{O}$

$LCA_{\mathcal{O}}$

$$K(\mathcal{O}) \to K(\mathbb{R})^r \oplus K(\mathbb{C})^s$$

$$\to K(LCA_{\mathcal{O}})$$

Lemma A, B Hausdorff TA

Say A is compactly generated.

$$\underline{\mathrm{Hom}}(\underline{A},\underline{B}) \cong \underline{\mathrm{Hom}(A,B)}$$

Pf)

$$\underline{\mathrm{Hom}(A,B)}(T)$$

$$= \mathrm{Hom}_{cts}\left(T, \mathrm{Hom}_{co}(A,B)\right)$$

$$\simeq \mathrm{Hom}_{cts}(T \times A, B)$$

$$\simeq \mathrm{Hom}_{Cond}(\underline{A} \times T, B)$$

$$\simeq \mathrm{Hom}(\mathbb{Z}[\underline{A} \times T], B)$$

$$
\begin{tikzcd}
\mathbb{Z}[\underline{A} \times T] = \mathbb{Z}[\underline{A}] \otimes \mathbb{Z}[T] \arrow[d] \\
A \otimes \mathbb{Z}[T] \\
\mathbb{Z}[\underline{A}] \arrow[r] & \underline{A}
\end{tikzcd}
$$

epi

(Inj.)

$$\underline{\mathrm{Hom}}(\underline{A},\underline{B})(T) = \mathrm{Hom}(\underline{A}\otimes\mathbb{Z}[T],\underline{B})$$

$$\downarrow$$

$$\underline{\mathrm{Hom}(A,B)}(T) = \mathrm{Hom}(\mathbb{Z}[A\times T], B)$$

- $\mathrm{Hom}_{Ab(Cond)}(\mathbb{Z}[\underline{A}\times T], \underline{B})$

$\simeq \mathrm{Hom}_{Cond}(\underline{A}\times T, \underline{B})$

$\simeq \mathrm{Hom}_{Cts}(T\times A, B) \simeq \mathrm{Hom}(T, \mathrm{Hom}(A,B))$

(Surj.)

- $\mathbb{Z}[\underline{A}\times\underline{A}] \to \mathbb{Z}[\underline{A}] \twoheadrightarrow \underline{A} \twoheadrightarrow 0$

$$\ker = \{[a+b]-[a]-[b]\}$$

- $T \longrightarrow \mathrm{Hom}(A,B)$

Want:
$$\begin{tikzcd}
\mathbb{Z}[\underline{A}]\otimes\mathbb{Z}[T] \arrow[r] \arrow[d] & \underline{B} \\
\mathbb{Z}[\underline{A}]\otimes T \arrow[ur, dashed, "?"'] &
\end{tikzcd}$$

$\Longleftrightarrow$
$$\begin{tikzcd}
\mathbb{Z}[A\times A]\otimes\mathbb{Z}[T] \arrow[d] \arrow[dr, bend left, "0"] & \\
\mathbb{Z}[A]\otimes\mathbb{Z}[T] \arrow[r] & \underline{B}
\end{tikzcd}$$

$$\iff \begin{tikzcd} A \times A \times T \arrow[r, "0"] \arrow[d] & B \\ A \arrow[ur, "f"'] & \end{tikzcd}$$

$$\iff \begin{tikzcd} A \times A \times t \arrow[r, "0"] \arrow[d] & B \\ A \arrow[ur, "f"'] & \end{tikzcd}$$

$\square$

Write $R\underline{\mathrm{Hom}}(A, B)$.

To proceed:

Thm (EM, Breen, Deligne.) $A \in Ab$

- $C^{con}_{\bullet}(A)$

with

- functorial $(Ab \to K_{\geq 0}(Ab))$
- $C^{con}_{n}(A) = \bigoplus_{\text{finite}} \mathbb{Z}[A^{\times \text{finite}}]$.

Canonical resolution

$$\begin{tikzcd}
\bigoplus_{j=1}^{n_i} \mathbb{Z}[A^{r_{i,j}}] \arrow[d, phantom, "\vdots"] \\
{} \arrow[d] \\
\mathbb{Z}[A^3] \oplus \mathbb{Z}[A^2] \arrow[d] \\
\mathbb{Z}[A^2] \arrow[d] \\
\mathbb{Z}[A] \arrow[d] \\
A
\end{tikzcd}$$

Pf)

[Step 1] $A = \mathbb{Z}^{\oplus n} = P \in \mathrm{Latt}$.

If ok in this case:

$C^{\mathrm{can}}_{\bullet}(P)$: $\cdots \to \mathbb{Z}[P^{\times}] \to \mathbb{Z}[P] \to P$

$$\begin{tikzcd}
\mathrm{Latt} \arrow[r] \arrow[d] & K(\mathrm{Ab}) \\
P_{\Sigma}(\mathrm{Latt}) \arrow[ur, dashed] &
\end{tikzcd}$$

$\underline{\simeq}$ Ab

[Step 2] WTS:

$C_\bullet^{con}(P)$ exists for PG Latt.

$\mathrm{Fun}(\mathrm{Latt}, \mathrm{Ab}) \leftarrow$ ab cat.

X is n-psuedo coh $\Leftrightarrow$

- $\mathrm{Ext}^i(X,-)$ compact for $i = 0, \ldots, n-1$

- length n <u>compact</u> <u>projectives</u>

$C_\bullet =$

Lemma $\to P_i \to \cdots \to P_0 \to X$

(not necc. a res.)

$P_i =$ cpct projectives

$H_i(C)$ is $n-i-1$

-pseudocoh for $i = 0, \ldots, n-1$

then X is n-pseudo coh.

[Step 3]. Induction step 1

$$\mathbb{Z}[p^2] \to \mathbb{Z}[p] \to P$$

$\mathbb{Z}[B^n P]$

M_j = f.g. free

$$H_i = \begin{cases} M_n \otimes_{\mathbb{Z}} P & i = 2n \\ \vdots & \\ M_1 \otimes_{\mathbb{Z}} P & i = n+1 \\ P & i = n \\ 0 & i = n-1 \\ \vdots & \\ 0 & i = 0 \end{cases}$$

- $\mathbb{Z}[B^n P] \longrightarrow P$
  $[-n]$

- $H_i = n-i-1$ pseudo coh. $i \in \{1, \ldots, n-1\}$

- $H_0 = n-1$ pseudo coh

$\Rightarrow$ done

$\blacksquare$

Lemma

1) $\underline{R\mathrm{Hom}}(\mathbb{R}, M) \simeq 0$.

2) $\underline{R\mathrm{Hom}}(\mathbb{T}, M) \simeq M[-1]$

Pf)

$$0 \to \underline{\underline{\mathbb{Z}}} \to \underline{\underline{\mathbb{R}}} \to \underline{\mathbb{T}} \to 0.$$

So 1) $\Rightarrow$ 2).

$T \in \mathrm{Cond.}$;

local to global S-S :

$$H^p_{\mathrm{Cond}}(T, \mathcal{E}xt^q(A, M)) \Rightarrow \underline{\mathrm{Ext}}^{p+q}(A, M)(T)$$

But

$$C^{can}_\bullet(A) \otimes \mathbb{Z}[S]$$

$$\wr$$

$$R\mathrm{Hom}(-, M)$$

$$\wr$$

$$\downarrow$$

$$H^2(A \times S; M) \longrightarrow \qquad \longrightarrow \prod_{j=1}^{k_n} H^2(A^{r_{k_n,j}} \times S; M)$$

$$H^1(A \times S; M) \longrightarrow \qquad \longrightarrow \prod_{j=1}^{k_n} H^1(A^{r_{k_n,j}} \times S; M)$$

$$\cdots$$

$$H^0(A \times S; M) \qquad \longrightarrow \prod_{j=1}^{n} H^0(A^{r_{k_n,j}} \times S; M)$$

$$\|$$

$$\Downarrow$$

$$\underline{\mathrm{Ext}}^{p+q}(A, M)(S)$$

- $0 \longrightarrow \mathbb{R}$

$$R\underline{\mathrm{Hom}}(\mathbb{R}, \mathbb{R}) \longrightarrow R\underline{\mathrm{Hom}}(0, \mathbb{R})_{12}^{0.}$$

- Claim boils down to

$$H^i_{\mathrm{sheaf}}(S \times \mathbb{R}; M) \cong H^i_{\mathrm{sheaf}}(S; M)$$

$\boxtimes$

Lemma

$$\varinjlim_{J \subseteq I} R\underline{\mathrm{Hom}}\left(\prod_J \pi, M\right)$$

$$\wr\!\!\mid$$

$$R\underline{\mathrm{Hom}}\left(\prod_I \pi, M\right).$$

Lemma

$$\underline{R\mathrm{Hom}}(A, \mathbb{R}) = 0$$

Pf)

$$C_\bullet^{con}(A) \otimes \mathbb{Z}[T]$$

$$\wr$$

$$\underline{R\mathrm{Hom}}(A, \mathbb{R})(T)$$

$$\wr \qquad C(T; \mathbb{R})$$

$$= \text{cts}$$

$\mathbb{R}$-valued f^{n}'s

$$\downarrow$$

$$
\underbrace{
\begin{array}{c}
0 \\
\downarrow \\
C(A \times T; \mathbb{R}) \\
\downarrow \\
C(A^{\times 2} \times T; \mathbb{R}) \\
\downarrow \\
C(A^{\times 2} \times T; \mathbb{R}) \oplus C(A^{\times 3} \times T; \mathbb{R}) \\
\downarrow \\
\vdots \\
\bigoplus_{\text{finite}} C(A^{\times \text{fin}} \times T; \mathbb{R}) \\
\downarrow \\
\vdots
\end{array}
}_{C^{\bullet}(A,T)}
$$

Claim:

$$2_{\mathbb{R}} : C^\bullet(A,T) \circlearrowleft$$

$$[2]_A \circlearrowright$$

are homotopic via $h^\bullet$

Assuming claim:

$$\begin{array}{c} f \\ \rotatebox{90}{$\in$} \\ C^i(A,T) \end{array} , \quad df = 0$$

$$2f = [2]^*(f) + d(h_{i-1}^*(f))$$

$$\Downarrow$$

$$f = \frac{1}{2}[2]^*(f) + d(h_{i-1}^*(f))$$

$$= \frac{1}{2^n}[2^n]^*(f) + d\Big(\frac{1}{2}h_{i-1}^*(f)$$

$$+ \frac{1}{4}h_{i-1}^*([2]^*(f)) + \cdots$$

$$+ \frac{1}{2^n}h_{i-1}^*\big([2^{n-1}]^*(f)\big)\Big).$$

- $[2^n]^* f$ has bdd norm,

$$\| [2^n]^* f \| \leq \| f \|$$

$\Rightarrow$

$$f = d\left(\frac{1}{2} h_{i-1}^*(f) + \frac{1}{4} h_{i-1}^*([2]^*(f)) + \cdots \right).$$

To summarize

domain.

	$\mathbb{T}$	$\mathbb{R}$	$\mathbb{Z}$
$\mathbb{T}$	$\mathbb{Z}[-1]$	$\mathbb{T} \to M$.	$\mathbb{T}$
$\mathbb{R}$	0	$\mathbb{R}$	$\mathbb{R}$
$\mathbb{Z}$	$\mathbb{Z}[-1]$	0	$\mathbb{Z}$

codomain.

Thm.

$$\underbrace{D^b(LCA)}_{\text{bounded derived category of LCA's.}} \hookrightarrow D(\mathrm{CondAb})$$

- agrees w/ Spitzweck-Hoffman theory.

- projective dimension 1

$$\underbrace{\hspace{8cm}}_{\text{compare w/ valuation rings of finite rank}}$$