§2 : positive results for when $N' = \tilde{N}'$ and other "fringe cases"

Recall the Coniveau S.S.

X dim d scheme $/k$.

$$\vec{Z} = \emptyset \subsetneq Z_d \subsetneq Z_{d-1} \subsetneq \cdots \subsetneq Z_0 = X$$

"complete flag" of closed subsch's.

$$\begin{tikzcd}
\to H^{p+q}_{Z_{p+1}}(X;A) \arrow[r] & H^{p+q}_{Z_p}(X;A) \arrow[r] & H^{p+q}_{Z_p \setminus Z_{p+1}}(X \setminus Z_{p+1};A) \arrow[dll, "+1"'] \\
H^{p+q+1}_{Z_{p+1}}(X;A) \to & &
\end{tikzcd}$$

$(\cdot)$

we get exact couple. $\rightsquigarrow$ S.S. $E^{p,q}_{\vec{Z}}$

$$F^p = \mathrm{Im}\left(H^n_{Z_p}(X;A) \longrightarrow H^n(X;A) \right)$$

these complete flags form a poset

$$\vec{Z}' \leq \vec{Z}$$

taking colim $E^{p,q}_{\vec{Z}}$ along this poset

gives us the coniveau S.S.

$E_1^{p,q}$

$$\begin{array}{ccccc}
 & & \vdots & & \\
\bigoplus_{x\in X^{(0)}} H^1_x(X;A) & \longrightarrow & \bigoplus_{x\in X^{(1)}} H^2_x(X;A) & \longrightarrow & \bigoplus_{x\in X^{(2)}} H^3_x(X;A) \\
\bigoplus_{x\in X^{(0)}} H^0_x(X;A) & \longrightarrow & \bigoplus_{x\in X^{(1)}} H^1_x(X;A) & \longrightarrow & \bigoplus_{x\in X^{(2)}} H^2_x(X;A)
\end{array}$$

$\uparrow$ cohomological

Gersten cplx.

local coh. of x

$\longrightarrow$ codimension

$X^{(j)}$ = set codim j pts of X

$$\mathrm{Fil}^p = \mathrm{Im}\Big(\sum_{\substack{Z \hookrightarrow X \\ \text{codim } p}} H^n(X;A) \longrightarrow H^n(X;A) \Big)$$

$$= N^p H^n(X;A) \qquad n = p+q.$$

When A is nice. ($k = \mathbb{C}$, X = smooth and A = constant sheaf of ab gp's).

Bloch–Ogus–Gabber computes the E_2-page:

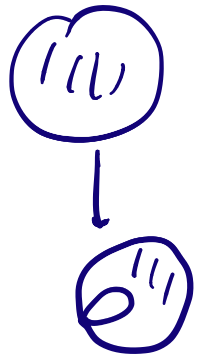
$$E_2^{p,q} = H^p(X; \mathcal{H}^q(A)) \Rightarrow H^n(X;A)$$

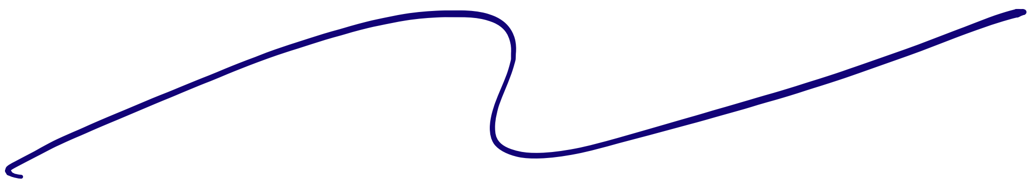
$\mathcal{H}^q(A)$: sheafification of $U \mapsto H^q(A)$.

$$H^p(\mathcal{H}^q) = H^0(X; \mathcal{H}^q(A))$$

E_2-page:

$$\begin{tikzcd}
H^0(\mathcal{H}^2) & & \\
H^0(\mathcal{H}^1) \arrow[drr, "d_2"] & \cdots & \\
H^0(\mathcal{H}^0) & H^1(\mathcal{H}^0) & H^2(\mathcal{H}^0).
\end{tikzcd}$$





$$\tilde{N}^c H^l(X;A) \hookrightarrow N^c H^l(X;A) \longrightarrow BO^{c,l}(X;A)$$

<u>Thm</u> 2.6 & 2.8.

I) $BO^{c,l}(-;A)$ is a stable birational invariant.

II) any torsion class is coniveau ≥ 1.

$\dim_{\mathbb{C}} X = d \quad \dim_{\mathbb{R}} X = 2d.$

$$H^{2d}(X;\mathbb{Z}) \cong \mathbb{Z}\{[X]\} \supseteq N^1 = 0 \text{ --- } 0 \text{ ---}$$

$H^{2d-1}(X;\mathbb{Z}) \supseteq N^1$

N^1 may not be zero. Take $\mathbb{C}$-manifold

w/ $\pi_1 X = \mathbb{Z}/2$ say. then by Thm II.

all torsion classes are coniveau ≥ 1 so by

P.D. N^1 not nec. $= 0$.

$H^1(X;\mathbb{Z}) \supseteq N^1$

$N^1 H^1(X;\mathbb{Z})$ is obtained by pushforward

from $H^1_Z(X;\mathbb{Z}) \xrightarrow{i_*} H^1(X;\mathbb{Z})$ $Z = \mathrm{codim}_{\mathbb{C}} 1$

$\cong H^1(Th(N_Z);\mathbb{Z})$

if Z smooth.

$\cong H^{1-2}(Z;\mathbb{Z}) = 0.$

<u>Lemma</u>. 2.2.I X is a smooth $\mathbb{C}$-variety

and A any abelian group.

then - .

then $$N^c H^\ell(X;A) = 0. \qquad \ell < 2c.$$

& $\underline{N^c H^{2c}}(X;A) =$ algebraic classes

- $N^1 H^2(X;\mathbb{Z})$ = classes in $H^2(X;\mathbb{Z})$ pushed forward from $\mathbb{C}$-codim 1-subvarieties.

$$\begin{array}{c} N^1H^2(X;\mathbb{Z}) \\ \uparrow\!\!\uparrow \\ CH^1(X;\mathbb{Z}) \end{array}$$

Conclusion: in this range,

$$N^c H^\ell = \tilde{N}^c H^\ell$$

for $\ell \le 2c$.

Lemma 2.2.I If $d = \dim_{\mathbb{C}} X$ and $d \le \ell - c$, then

$$\tilde{N}^c H^\ell(X;A) = N^c H^\ell(X;A).$$

Pf by examples

$$\left.\begin{array}{l} c=1 \\ l=5 \\ d=4. \end{array}\right\}$$ divisors on a 4-fold, coh. deg $=5$.

$$4 \leq 5-1.$$

$$H^{5-2}(\underline{D};\mathbb{Z}) \longrightarrow H^5(X;\mathbb{Z})$$

smooth.

$D \hookrightarrow X$ a divisor.

Say $X \hookrightarrow \mathbb{CP}^N$, $D = X \cap H$

H hyp. plane $\mathbb{CP}^N$. chosen s.t. D smooth (by Bertini.)

$\blacksquare$

$\tilde{N} = N.$

Pf of main thm).

WTS: $BO^{1,l}(-;A)$ $\forall l \in \mathbb{N}.$, any A.

this is a proper birational inv.

$$V^\ell(X) := \mathrm{coker}\left(CH^\ell(X) \longrightarrow H^{\ell\ell}(X)\right)$$

birational inv of proper smooths if

$\ell = 0, 1$ or $\dim X$, $\underline{\dim}\ X-1$
$= 2.$

$\underline{\text{basic idea of Pf}}$) IHC is known for $\ell = \dim X, 0,$
1 by Lefschetz (1,1)

$$\underset{\text{SmProj}}{X'} \dashrightarrow X \quad \text{birational equiv}$$

then by R.O.S. + weak fact

bl up, bl down smooth center

$$\begin{tikzcd} & \cdot \arrow[dl] \arrow[dr] & \\ X' \arrow[rr, dotted] & & X \end{tikzcd}$$

So suffice to show that if $X \in \mathrm{SmProj}$

and $\underset{\mathrm{Sm}}{Z} \hookrightarrow \underset{\mathrm{Sm.}}{X}$ $\quad \tilde{X} = Bl_Z X \longrightarrow X$

then $V^\ell(X) = V^\ell(\tilde{X})$ in this range.

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d] & \tilde{X} \arrow[d] \\ Z \arrow[r, hook] & X \end{tikzcd}$$

lower dimensions

$\rightsquigarrow$ gets taken by CH and by H^{tot} to appropriate LES.

Step I $\quad X \times \mathbb{P}^n \xrightarrow{pr} X.$

then consider.

$$BO^{1,\ell}(X \times \mathbb{P}^n; A) \cong \frac{pr^* N^1 H^\ell(X;A)}{\tilde{N}^1 H^\ell(X \times \mathbb{P}^n; A)}.$$

✓ Künneth.

Claim: $\alpha \in H^\ell(X;A)$ coniveau 1

then tfae:

1) $pr^*\alpha$ is strong con. 1.

2) α is ~~to~~ strong con. 1.

2) $\Rightarrow$ 1) ✓.

to prove 1) $\Rightarrow$ 2).

Since $pr^*\alpha$ is strong con. 1. $\Leftrightarrow$ $\exists$

rel dim -1

$n/$

$$f : V \longrightarrow X \times \mathbb{P}^n$$

so $V = \dim X + r - 1.$

such that

$$\boxed{pr^*\alpha = f_*\beta}$$

$$\beta \in H^{\ell-2(2-1)}(V; A)$$

now for a section $i : X \hookrightarrow X \times \mathbb{P}^n$, form the pullback:

$$\begin{tikzcd} W \arrow[r, "i'"] \arrow[d, "g"'] & V \arrow[d, "f"] \\ X \arrow[r, hook, "i"'] & X \times \mathbb{P}^n \end{tikzcd}$$

but now using base change + fact that $pr\, i = id.$ we see that

$\alpha = g_*(i'^*\beta)$. So choosing a section i such that W is smooth does the job.

__Step II.__ It suffices to show using R.O.S + weak factorization that if $\tilde{X} \longrightarrow X$ blowup along smooth center, that $BO^{1,l}(\tilde{X}, A) \cong BO^{1,l}(X; A)$.

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d] & \tilde{X} \arrow[d, "\pi"] \\ \underset{\text{smooth}}{Z} \arrow[r, hook] & X \end{tikzcd}$$

by the LES for

$$H^l(\tilde{X}; A) = \{\pi^* H^l(X; A) \text{ and } \underbrace{H^l_E(\tilde{X}; A)}_{\text{always strong coniveau 1 since } E \xrightarrow{\text{cod } 1} X,\ \underset{\text{smooth}}{E}}\}.$$

So bir. inv. follows as soon as we prove the following claim:

$\alpha \in H^{\ell}(X; A)$ then.

$\pi^*\alpha$ has (strong) coniveau 1
iff
α has (strong) coniveau 1.

Pf of claim.

$$\pi_*: H^{\ell}(\tilde{X}; A) \longrightarrow H^{\ell}(X; A).$$

$$\pi_* \pi^* \alpha = \alpha.$$

then claim follows from correspondences:

$\Gamma \in CH^d(X \times Y)$ X, Y dim d.

then

$$\Gamma_* = H^{\ell}(Y; A) \longrightarrow H^{\ell}(X; A)$$

$\|$

(graph π)

$\cap$

$X \times \tilde{X}$

pres. coniveau ≥ 1
and
strong coniveau ≥ 1.

· for coniveau ≥ 1 this comes from the Bloch-Ogus S.S.

$N^1 = \mathrm{Fil}^1 \stackrel{?}{=} \ker\left(H^q(X;A) \to H^0(X; \mathcal{H}^q)\right).$

- Γ_* acts on the whole S.S. and pres the filtration.

Strong coniveau. ! X smooth proper

$$\tilde{N}^c H^q(X;A) = \sum \mathrm{Im}\, \Gamma_*$$

$\tilde{\Gamma} \to \Gamma \in CH^k(X \times Y)$

Y smooth proper dim $k-r$

$r \geq c$.

$\therefore$ Since $BO^{1,*}(-;A)$

Stable under $\mathbb{P}^n$ and bir ~~[illegible]~~ equiv

$\Rightarrow BO^{1,*}(-;A)$ Stable bir inv. $\blacksquare$

Thm $\begin{pmatrix} \text{CTV,} \\ \text{B.S,} \\ \text{B} \end{pmatrix}$ any torsion class is coniveau ≥ 1.

Pf) By above discussion on coniveau

(2).

$$N^1 H^\ell(X;\mathbb{Z}) = \ker\left(H^\ell(X;\mathbb{Z}) \longrightarrow H^0(X; \mathcal{H}^\ell)\right)$$

$\overset{\shortparallel}{\text{gr}^0.}$

Enough to prove that

is in fact torsion free.

In fact the sheaf $\mathcal{H}^\ell(\mathbb{Z})$ is torsion free.

SES of ab shvs.

$$0 \to \mathbb{Z}(i) \xrightarrow{\times n} \mathbb{Z}(i) \longrightarrow \mu_n^{\otimes i} \to 0$$

$$(\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/n)$$

LES of ~~groups~~ ab sheaves

$$\to \mathcal{H}^{p-1}(\mu_n^{\otimes i}) \xrightarrow{0} \mathcal{H}^p_X(\mathbb{Z}(i)) \xrightarrow{\times n} \mathcal{H}^p_X(\mathbb{Z}(i))$$

$$\to \mathcal{H}^p_X(\mu_n^{\otimes i}) \xrightarrow{+1}$$

NTS: $\mathcal{H}^{p-1}(\mathbb{Z}(i)) \longrightarrow \mathcal{H}^{p-1}(\mu_n^{\otimes i})$

$$\mathcal{H}^2(\mathbb{Z}) \xrightarrow{f} \mathcal{H}^2(\mathbb{Z}/n)$$

surjective.

$\pi_*: X_{an} \longrightarrow X_{Zar}$

$\pi_* \mathcal{O}_{an}^\times \longrightarrow \mathcal{H}^1_{\mathbb{Z}}$ from

the exp seq.

Claim f surj

$$\begin{tikzcd}
\mathcal{O}_X^\times|^{\otimes 2} \arrow[r] \arrow[d, "\text{symbol}"] & (\pi_* \mathcal{O}_{an}^\times)^{\otimes 2} \arrow[r] & \mathcal{H}^1 \otimes \mathcal{H}^1 \arrow[r] & \mathcal{H}^2(\mathbb{Z}) \arrow[d] \\
K_{2,X} \arrow[r] \arrow[rrr, bend right, "\text{norm residue map}"'] & K_{2,X} \otimes \mathbb{Z}/n \arrow[rr, "\substack{\text{S-M.}\\(\text{V.})\\ \cong}"] & & \mathcal{H}^2(\mathbb{Z}/N)
\end{tikzcd}$$

$\uparrow$ sheafification of K_2.

D

$$CH^{d-c}(Z) \to CH^d(X) \to CH^d(\bullet) \to 0$$

Spec $k(X)$

$U \hookrightarrow X$

$d?//$