GAGA-Type conjecture for the Brauer group via derived geometry

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Introduction / Motivation:

X q.c.q.s scheme.

$$\begin{array}{c} Br(X) := H^2_{ét}(X, \mathbb{G}_m) \quad \text{Brauer-Grothendieck group of } X \\ \cup\!\!| \\ H^2_{ét}(X, \mathbb{G}_m)_{tors} \\ \text{Skolem-Noether} \Longrightarrow \uparrow \\ Br_{Az}(X) \end{array}$$

$\mathcal{A}$ Azumaya algebra: $\mathcal{A} \in Alg(QCoh(X))$ such that

(1) $\mathcal{A}$ is a locally free $\mathcal{O}_X$-Module of finite rank

(2) $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A}^{op} \xrightarrow{\sim} \underline{End}(\mathcal{A})$

The (local) Brauer–Manin pairing:

R complete DVR
$K = \mathrm{Frac}(R)$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ K \arrow[r] & R \end{tikzcd}$$

X/K smooth & proj.
$\mathcal{X}/R$ regular, flat (with geom. integral fibers)

$$\text{(BM)} \qquad CH_0(X)^0 \times \frac{Br(X)}{Br(K)+Br(\mathcal{X})} \to \mathbb{Q}/\mathbb{Z}$$

$$\downarrow Alb_X \qquad\qquad \uparrow \varphi$$

$$\text{(Tate)} \qquad Alb_X(K) \times H^1(K, Pic^0(\bar{X})) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

Compatible pairing (Lichtenbaum 1969, Kai 2015)

& $\mathrm{Coker}(Alb_X) \simeq (\mathrm{Ker}\,\varphi)^*$
$\simeq \mathrm{Coker}(Pic(\bar{X})^{G_K} \to NS(\bar{X})^{G_K})$

Provided that $\mathcal{X}/R$ is SMOOTH & $\underline{Pic}_{\mathcal{X}/R}$ is smooth.

The condition on $\underline{\mathrm{Pic}}_{X/R}$ is difficult to achieve:
How does it appear? This is related to the contribution of $\mathrm{Br}(X)$

In fact we have 3 players that we can attach to X/R:

(1) $\mathrm{Br}(X) = H^2_{\text{ét}}(X, \mathbb{G}_m)$

(2) $\mathrm{Br}(\mathfrak{X} \to \mathrm{Spf}(R)) := H^2(\operatorname{holim}_n (\tau_{\leq 2} R\Gamma(\mathfrak{X}_n, \mathbb{G}_m)))$

where $\mathfrak{X}_n = X \otimes_R R/\mathfrak{m}^n$, $\mathfrak{X} = \operatorname{colim} \mathfrak{X}_n$

(3) $\varprojlim_n \mathrm{Br}(\mathfrak{X}_n)$

Have a factorization

$$\begin{tikzcd}
\mathrm{Br}(X) \arrow[r] \arrow[rr, bend right, "(\bigstar)"'] & \mathrm{Br}(\mathfrak{X}) \arrow[r] & \lim_n \mathrm{Br}(\mathfrak{X}_n)
\end{tikzcd}$$

Grothendieck (Brauer III):

* $\mathcal{X}/_R$ proper flat, $\mathcal{X}$ regular, R Henselian excellent DVR
* Assume $\lim^1_n \mathrm{Pic}(\mathcal{X}_n) = 0$

Then $(\star)$: $\mathrm{Br}(\mathcal{X}) \to \lim_n \mathrm{Br}(\mathcal{X}_n)$ is injective

<u>Remarks</u>:

Trivial Azumaya algebra

(1) $\mathcal{X}$ regular $\Rightarrow \mathrm{Br}_{Az}(\mathcal{X}) = \mathrm{Br}(\mathcal{X})$

$\mathrm{Ker}(\star) \ni \mathcal{A} \in \mathrm{Br}_{Az}(\mathcal{X}) \mapsto \{\mathcal{A}_n := \mathcal{A} \otimes_R R/m^n \xrightarrow[\sim]{\sigma_n} \underline{\mathrm{End}}(\mathcal{E}_n)\}$

Then use $\lim^1$ vanishing to lift $\{\mathcal{E}_n\}$ to $\hat{\mathcal{E}}/_{\mathcal{X}}$ + $\sigma: \mathcal{A} \xrightarrow{\sim} \underline{\mathrm{End}}(\hat{\mathcal{E}})$

(2) When does $\lim^1_n \mathrm{Pic}(\mathcal{X}_n) = 0$ hold?

For example if $\mathrm{Pic}(\mathcal{X}_{n+1}) \twoheadrightarrow \mathrm{Pic}(\mathcal{X}_n) \Leftarrow \underline{\mathrm{Pic}}_{\mathcal{X}/R}$ smooth

Question (Brauer III, Rem. 3.4):

Is the map $(\star)$ injective on $\mathrm{Br}_{Az}(X)$ for every X/R proper with R Henselian local?

How to study this problem?

Step 1. Milnor sequence

$$
\begin{tikzcd}
 & & & \lim\limits_{n} \mathrm{Br}(X_n) \arrow[d, equal] & \\
0 \arrow[r] & \underbrace{\lim\limits_{n}{}^{1} H^1(X_n, \mathbb{G}_m)}_{= \lim^1 \mathrm{Pic}(X_n)} \arrow[r] & \mathrm{Br}(\mathfrak{X}) \arrow[r] & \lim\limits_{n} H^2(X_n, \mathbb{G}_m) \arrow[r] & 0 \\
 & & \mathrm{Br}(X) \arrow[u, "(\star\star)"] \arrow[ur, "(\star)"'] & & \\
 & & \mathrm{Br}_{Az}(X) \arrow[u, phantom, sloped, "\subseteq"] \arrow[uur, bend right, "(\star\star\star)"'] & &
\end{tikzcd}
$$

What about $(\star\star)$?

We claim that it is always injective.
(while $(\star\star\star)$ is NOT)

Step 2: Derived Morita theory (after Töen)

Recall: (classical Morita theory) R commutative ring, $A, B \in \mathrm{Alg}_R$

* $A \underset{\text{Morita}}{\sim} B \overset{\text{def}}{\Longleftrightarrow} \mathrm{Mod}_A \cong \mathrm{Mod}_B$
* $\mathrm{Fun}_c(\mathrm{Mod}_A, \mathrm{Mod}_B) \cong A^{op} \otimes_R B$ - Modules
* A is Azumaya $\Longleftrightarrow$ A is invertible up to Morita $\Leftrightarrow \exists A' : A \otimes_R A' \underset{\text{Morita}}{\sim} R$

Derived version: R commutative dg-algebra

Mod_R = Stable ∞-Cat of R-Modules $\supset \mathrm{Mod}_R^{\omega} = \mathrm{Perf}(R)$

$\mathrm{Cat}_R^{\otimes}$ = R-linear (small) ∞-Cat.

$\mathcal{C}, \mathcal{C}' \in \mathrm{Cat}_R^{\otimes}$, $\mathcal{C} \underset{\text{Morita}}{\sim} \mathcal{C}'$ if $\overbrace{\mathrm{Fun}(\mathcal{C}, \mathrm{Mod}_R)}^{\hat{\mathcal{C}}} \cong \overbrace{\mathrm{Fun}(\mathcal{C}', \mathrm{Mod}_R)}^{\hat{\mathcal{C}}'}$

in Pr_R^L

= Presentable stable R-linear ∞-Categories.

$\mathrm{Cat}^{\otimes}_{R, \sim \text{Morita}} \doteq$ Localization of $\mathrm{Cat}_R^{\otimes}$ to Morita equivalences.

Thm. (Tabuada, Töen, Robalo, Antieu-Gepner...)

$$\begin{tikzcd} \mathrm{Cat}_R \arrow[r, "(-)^\wedge"] \arrow[d, shift right] & \mathrm{Pr}^L_R = \mathrm{Mod}_{\mathrm{Mod}_R}(\mathrm{Pr}^L) \supseteq \mathrm{Pr}^{L,cg}_R \\ \mathrm{Cat}^{idem}_R \arrow[u, shift right] \arrow[r, shift left, "\mathrm{Ind}", "\sim"'] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{Pr}^{L,\omega}_R \arrow[u] \arrow[l, shift left, "(-)^\omega"] \\ \mathrm{Cat}_{R,\sim \mathrm{Morita}} & \end{tikzcd}$$

Not fully faithful

f.f.

$= $ comp. gen. pres. R-linear, stable ∞-Cat.

closed symmm. monoidal ∞-Cat

$\mathcal{C}, \mathcal{C}' \in \mathrm{Pr}^{L,\omega}_R$, $\mathcal{C} \otimes \mathcal{C}'$ Represents bifunctors $\mathcal{C} \times \mathcal{C}' \to \mathrm{Mod}_R$ that commute with compact objects and filtered colimits in both variables.

Duality:

$\mathcal{C} \in \mathrm{Pr}^{L,\omega}_R$

$$D_R(\mathcal{C}) = \mathrm{Fun}^L_R(\mathcal{C}, \mathrm{Mod}_R)$$

$$\mathrm{ev}: \quad \mathcal{C} \otimes D_R(\mathcal{C}) \to \mathrm{Mod}_R$$

$$\mathrm{coev}: \quad \mathrm{Mod}_R \to \mathcal{C} \otimes D_R(\mathcal{C})$$

$$(\mathcal{C} \in \mathcal{C} \otimes_R D_R(\mathcal{C})\text{-Mod})$$

A priori maps only in Pr^L_R, Not in $\mathrm{Pr}^{L,\omega}_R$

+ usual combatibility.

Def: $\mathcal{C} \in \mathrm{Pr}^{L,\omega}_R$ is proper if ev is in $\mathrm{Pr}^{L,\omega}_R$ ($\mathcal{C}(x,y)$ cpt $\forall x,y$)
is smooth if (dualizable in Pr^L_R &) coev is in $\mathrm{Pr}^{L,\omega}_R$

Rmk: $\mathcal{C}$ smooth $\Longrightarrow$ $\mathcal{C} \simeq \mathrm{Mod}_A$ for some $A \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Mod}_R)$

So: $\mathcal{C} \in \mathrm{Pr}^{L,\omega}_R$ dualizable $\Longleftrightarrow$ $\mathcal{C}$ smooth & proper

Azumaya algebras?

(1) $B \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Mod}_R)$, get an object in $\mathrm{Cat}^{\otimes}_R \longrightarrow \mathrm{Cat}^{\otimes}_{R,\sim \mathrm{Morita}} \simeq \mathrm{Pr}^{L,\omega}_R$
(cat. with 1 object) $\quad [B]$

(2) $A \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Mod}_R)$ is Azumaya $\Longleftrightarrow$ $[A]$ is invertible
$\overset{\text{Def}}{\Longleftrightarrow}$ (a) A cpt gen for Mod_R
(b) $A \overset{\mathbb{L}}{\otimes} A^{op} \xrightarrow{\simeq} \mathrm{End}(A)$ q.iso

Thm (Töen): $\mathcal{C} \in \mathrm{Pr}^{L,\omega}_R$ invertible $\Longleftrightarrow$ $\mathcal{C} \simeq \mathrm{Mod}_A$ with A Azumaya

The notion globalizes in a non trivial way:

Consider $\mathrm{QCoh}^{cat} : \mathrm{dAff}_R^{op} \to \mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\otimes})$

$\mathrm{Spec}(A) \mapsto \mathrm{Pr}^L_A = \mathrm{Mod}_{\mathrm{Mod}_A}(\mathrm{Pr}^L)$

and the subfunctors $\mathrm{QCoh}^{cat}_{cg}\ (\mathrm{Spec}(A) \mapsto \mathrm{Pr}^{L,cg}_A \subseteq \mathrm{Pr}^L_A)$

$\mathrm{QCoh}^{cat}_{\omega}\ (\mathrm{Spec}(A) \mapsto \mathrm{Pr}^{L,\omega}_A)$

Thm. QCoh^{cat}, QCoh^{cat}_{cg} satisfy étale descent (Lurie)

$\mathrm{QCoh}^{cat}_{\omega}$ satisfies Zariski descent

$X \in \mathrm{dSt}k/_R$, get comparison functor $\Gamma_X : \mathrm{QCoh}^{cat}(X) \to \mathrm{Mod}_{\mathrm{QCoh}(X)}(\mathrm{Pr}^L_R)$

Def. X is 1-affine if Γ_X is an equivalence

Thm. (Gaitsgory) X qcqs derived scheme is 1-affine.

In fact, more is true:

If X qcqs derived scheme, we have another functor

$$\Gamma_X^\omega : \mathrm{QCoh}_\omega^{cat}(X) \to \mathrm{Mod}_{\mathrm{QCoh}(X)}(\mathrm{Pr}_R^{L,\omega}) =: NC_X$$

Prop. (B.-P.)

- (1) Γ_X^ω is an equivalence
- (2) $\mathcal{C} \in NC_X$ is dualizable iff $\exists$ Affine Zariski open cover $\{U_i = \mathrm{Spec}(A_i)\}_i$ such that $\mathcal{C} \otimes_{\mathrm{QCoh}(X)} \mathrm{QCoh}(U_i) \in \mathrm{Pr}_{A_i}^{L,\omega}$ is smooth & proper.
- (3) $\mathcal{C} \in NC_X$ is invertible iff $\exists$ Affine Zariski open cover $\{U_i = \mathrm{Spec}(A_i)\}_i$ such that $\mathcal{C} \otimes_{\mathrm{QCoh}(X)} \mathrm{QCoh}(U_i) \in \mathrm{Pr}_{A_i}^{L,\omega}$ is Morita equivalent to a derived Azumaya alg.

Moreover: $\mathcal{C} \in NC_X^{sm \,\&\, prop} \iff \mathcal{C} \simeq \mathrm{Mod}_{\mathcal{A}}(\mathrm{QCoh}(X))$

$\mathcal{A}$ sheaf of perfect $\mathcal{O}_X$-Mod.

Thm (B.-P.) (R, m) Noetherian complete local ring

X proper derived scheme $/R$.

Then the symmetric monoidal functor

$$\nu_\bullet : NC_X^{sm \,\&\, prop} \longrightarrow \lim_n NC_{X_n}^{sm \,\&\, prop}$$

$$\mathcal{C} \mapsto \{\mathcal{C}_n\}$$

is fully faithful

$\mathrm{Mod}_{\mathrm{QCoh}(X)}(\mathrm{Pr}_R^{L,\omega})$ $\|$

Equivalently, $\forall\, \mathcal{C} \in NC_X^{sm \,\&\, pr.}$, $\mathcal{C} \xrightarrow{\sim} \lim_n \mathcal{C}_n$ in NC_X

Tool for the proof: $\mathcal{C}^{\omega} \simeq \mathrm{Mod}_{\mathcal{A}}(\mathrm{Perf}(X))$

Formal GAGA for perfect cpx.

Applications to the (derived) Brauer group:

$dAz^{cat}(X) :=$ ∞-cat of derived Azumaya alg. on X

$\subseteq Pr^{L,\omega}_X = NC_X$ (full sub. spanned by $\mathcal{A}$-Mod for $\mathcal{A}$ Azumaya)

Thm (Töen): X qcqs derived scheme

* $dAz^{cat}(X) \cong NC^{inv}_X \subset NC_X^{sm \,\&\, prop}$
* $dAz \simeq K(\mathbb{G}_m, 2) \times K(\mathbb{Z}, 1)$ as derived stacks
* $dBr(X) \cong \pi_0(dAz(X)) = H^2(X, \mathbb{G}_m) \times H^1(X, \mathbb{Z})$

$\uparrow$ globally defined Azumaya algebras$/X$

Formal injectivity:

X derived scheme, proper / $\mathrm{Spec}(R)$, R Noetherian, complete, local.

$\mathfrak{X} = \mathrm{colim}\ X \otimes_R R/m^n$

$$d\mathrm{Br}(X) \to d\mathrm{Br}(\mathfrak{X}) := \pi_0(\mathrm{Map}(\mathfrak{X}, d\mathrm{Az})) \to \lim_n H^2(X_n, \mathbb{G}_m)$$

Milnor sequence: $d\mathrm{Az}(\mathfrak{X}) = \lim_n d\mathrm{Az}(\mathfrak{X}_n)$

$$0 \to \underbrace{\lim{}^1 \pi_1(d\mathrm{Az}(\mathfrak{X}_n))}_{\substack{= \lim^1 \mathrm{Pic}(X_n) \\ + \\ 0 = \lim^1 H^0(X_n, \mathbb{Z})}} \to d\mathrm{Br}(\mathfrak{X}) \to \lim_n d\mathrm{Br}(X_n) \to 0$$

Cor: $NC_X^{\text{sm\&prop}} \hookrightarrow \lim_n NC_{X_n}^{\text{sm.p.}}$

$\uparrow$

$\pi_0(d\mathrm{Az}(X))$ (passing to inv. obj.)

Rmk: (1) The map is **NOT surj.** in general.

(enough to take X/R dim 1 with X regular, X_1 Nodal curve)

(2) R Henselian q.excellent also OK.

(3) if $\dim_R X \leq 1$, we get $H^2(X, \mathbb{G}_m) \hookrightarrow \lim_n H^2(X_n, \mathbb{G}_m)$

This is even stronger than what conjectured by Grothendieck!

(But this fails in dim ≥ 2) Cfr Brauer III, 3.4-(a).

Formal GAGA for twisted sheaves:

We can recover the formal injectivity from another point of view:

$$\begin{array}{ccccl}
\alpha & \in H^2(X, \mathbb{G}_m) & \leftrightarrow & \mathcal{A} \xrightarrow{\pi} X & \mathbb{G}_m\text{-gerbe (locally } X \times B\mathbb{G}_m) \\
\downarrow & \downarrow & & \downarrow & \\
0 & \in H^2(\mathfrak{X}, \mathbb{G}_m) & & \{\mathcal{A}_n \xrightarrow{\pi_n} X_n\}, & f_n : \mathcal{A}_n \xrightarrow{\simeq} B\mathbb{G}_m \times X \to B\mathbb{G}_m
\end{array}$$

Global trivialization $\Longleftarrow$ $\mathrm{Pic}(\mathcal{A}) \to \lim_n \mathrm{Pic}(\mathcal{A}_n)$

Rmk: $\mathcal{A}/X$ is a relative Artin stack, but NOT proper!

Thm (B.-P.) $\pi : \mathcal{X} \to X$ $\mathbb{G}_m$-gerbe, $X \to S$ proper derived scheme

$S = \mathrm{Spec}(R)$, R complete, Noetherian, local.

* $\mathrm{APerf}(\mathcal{X}) \xrightarrow{\sim} \lim_n \mathrm{APerf}(\mathcal{X}_n)$
* $\mathrm{Perf}(\mathcal{X}) \xrightarrow{\sim} \lim_n \mathrm{Perf}(\mathcal{X}_n)$

Proof: $$\mathrm{QCoh}(\mathcal{X}) \cong \prod_{\chi} \mathrm{QCoh}_{\chi}(\mathcal{X})$$

(derived version of a result of Lieblich).