# Harvard Thursday Seminar – Hermitian K-Theory

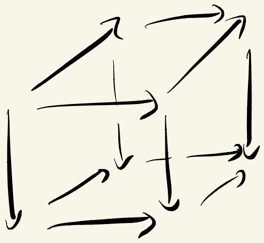
## Talk 1: Mind your $\mathcal{C}$'s & $\mathcal{Q}$'s

1. Quadratic, Bilinear, Linear
2. Recognition criteria
3. Adjunctions galore
4. The quadratic stable recollement
5. Hermitian & Poincaré

## 1] Quadratic, Bilinear, Linear

Recall: $F: \mathcal{C} \longrightarrow \mathcal{D}$ is $\boxed{2\text{-excisive}}$

if



$\longmapsto$ Cartesian cube

all faces cocart
(strongly cocart)

**Def**: Let $\mathcal{C}$ be stable & $\mathcal{Q}: \mathcal{C}^{op} \longrightarrow Sp$ functor.

We say that $\mathcal{Q}$ is:

- $\boxed{\text{Linear}}$ if reduced & $\boxed{1}$-excisive exact
- $\boxed{\text{Quadratic}}$ if reduced & $\boxed{2}$-excisive

# Guiding examples

- Linears : $\boxed{\varphi := \mathrm{Hom}_R(-, M)} : \mathrm{Mod}_R^{op} \longrightarrow \mathrm{Sp}$

- Quadratic : $\boxed{\varphi := \mathrm{Hom}_R(\cancel{X} \otimes \cancel{X}, M) \circ \Delta} : \mathrm{Mod}_R^{op} \longrightarrow \mathrm{Sp}$

# Dictionary

- 2 - exc $\longleftrightarrow$ $a + bx + cx^2$
- Quadratic $\longleftrightarrow$ $bx + cx^2$
- 1 - exc $\longleftrightarrow$ $a + bx$
- Linear $\longleftrightarrow$ $bx$

# Legend:

- $\boxed{\mathrm{Fun}_*(\mathcal{C}^{op} \times \mathcal{C}^{op}, \mathrm{Sp})}$ reduced bifunctors

  $\cup|$

- $\boxed{\mathrm{BiFun}(\mathcal{C})}$ bireduced bifunctors

  $B(x,y) \simeq 0$ if $x \simeq 0$ or $y \simeq 0$

  $\cup|$

- $\boxed{\mathrm{Fun}^b(\mathcal{C})}$ bilinear bifunctors

- $\boxed{\mathrm{Fun}^s(\mathcal{C}) := \mathrm{Fun}^b(\mathcal{C})^{hC_2}}$ sym bilinear bifunctor

- $\boxed{\mathrm{Fun}^q(\mathcal{C}) \subseteq \mathrm{Fun}_*(\mathcal{C}^{op}, \mathrm{Sp})}$ quadratic functor

  Note: closed under (co)limits, so stable too

## Cons:

$\beta \in \mathrm{Fun}^b(\mathcal{C})$

(1) Note that we have a retraction

[Bireduction] $\beta(X,0) \oplus \beta(0,Y) \longrightarrow \beta(X,Y) \longrightarrow \beta(X,0) \oplus \beta(0,Y)$

So taking (co)fibre gives

$$(-)^{red} : \mathrm{Fun}_*(\mathcal{C}^{op} \times \mathcal{C}^{op}, \mathcal{S}_p) \longrightarrow \mathrm{BiFun}(\mathcal{C})$$

Concretely, $\beta(x,y) \simeq \beta(X,0) \oplus \beta(0,Y) \oplus \beta(X,Y)^{red}$

Clearly commutes with arbitrary $\boxed{\text{(co)lims.}}$

(2) Can define

[Cross Effects] $B_{(-)} : \mathrm{Fun}^q(\mathcal{C}) \longrightarrow \mathrm{BiFun}(\mathcal{C})$

$$\varphi \longmapsto \varphi\left(\overset{X}{-} \oplus \overset{Y}{-}\right)^{red}$$

$B^{\Delta}_{(-)} := \boxed{\Delta^* \circ B_{(-)}} : \mathrm{Fun}^q(\mathcal{C}) \xrightarrow{B_{(-)}} \mathrm{BiFun}(\mathcal{C}) \xrightarrow{\Delta^*} \mathrm{Fun}^q(\mathcal{C})$

Have natural transformations

$$(B^{\Delta}_{\varphi} \Rightarrow \varphi)(X) = \left(\varphi(X \oplus X)^{red} \hookrightarrow \varphi(X \oplus X) \xrightarrow{\Delta^*} \varphi(X)\right)$$

$$(\varphi \Rightarrow B^{\Delta}_{\varphi}) = \left(\varphi(X) \xrightarrow{\nabla^*} \varphi(X \oplus X) \longrightarrow \varphi(X \oplus X)^{red}\right)$$

$$\rightsquigarrow \quad \boxed{(B^{\Delta}_{\varphi})_{hC_2} \Rightarrow \varphi \Rightarrow (B^{\Delta}_{\varphi})^{hC_2}}$$

**Obs**: For $\beta \in \mathrm{Fun}^b(\mathcal{C})$, easy to show that

$$B_{\tau\beta}(x,y) \simeq \beta(x,y) \oplus \beta(y,x)$$

(3) Have [Assoc Quadratics]

$$\boxed{\varphi^q_\beta := (\Delta^*\beta)_{hC_2}} : \mathrm{Fun}^s(\mathcal{C}) \longrightarrow \mathrm{Fun}^q(\mathcal{C})$$

$$\boxed{\varphi^s_\beta := (\beta^\Delta)^{hC_2}} : \mathrm{Fun}^s(\mathcal{C}) \longrightarrow \mathrm{Fun}^q(\mathcal{C})$$

They're quadratic HA.6.1.3.5 + $\mathrm{Fun}^q(\mathcal{C})$ closed (co)lims

<u>Obs:</u> $\boxed{B_{\varphi^q_\beta} \simeq \beta \simeq B_{\varphi^s_\beta}}$

$$B_{\varphi^q_\beta}(X,Y) = \big(\beta(X,Y) \oplus \beta(Y,X)\big)_{hC_2} \simeq \beta(X,Y) \simeq \cdots \simeq B_{\varphi^s_\beta}(X,Y).$$

(4) Define [(Co)linear, (Co)homogeneity]

$$(B^\Delta_\varphi)_{hC_2} \xRightarrow[\simeq]{\text{homogeneous}} \varphi \Rightarrow \boxed{L_\varphi} \quad \text{Linear part}$$

$$\boxed{cL_\varphi} \Rightarrow \varphi \xRightarrow[\text{(co)homogeneous}]{\simeq} (B^\Delta_\varphi)^{hC_2}$$

Colinear part

$\cancel{bx} + cx^2$

## 2) Recognition criteria

**Lem** (Total fibre yoga): Have can. equiv

$$\mathrm{tofib}\begin{pmatrix} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{pmatrix} \simeq \mathrm{fib}\big(A \to B \times_D C\big)$$

**Lem** (Quadratic failure of exactness):

Let $\varphi : \mathcal{C}^{op} \to Sp$ be quadratic & $B := B_\varphi$.

Suppose have
$$\begin{tikzcd} x \arrow[r, "a'"] \arrow[d, "b'"'] & y \arrow[d, "b"] \\ z \arrow[r, "a"'] & w \end{tikzcd}$$
exact in $\mathcal{C}$. Then

$$\begin{tikzcd} B(\mathrm{cofib}(a'), \mathrm{cofib}(b')) \arrow[r] \arrow[d, equal] & \varphi(w) \arrow[r] \arrow[d] & \varphi(z) \times_{\varphi(x)} \varphi(y) \arrow[d] \\ B(\mathrm{cofib}(a'), \mathrm{cofib}(b')) \arrow[r] & B(z,y) \arrow[r] & B(z,x) \times_{B(x,x)} B(x,y) \end{tikzcd} \quad (*)$$

$\Rightarrow$ $\varphi$ quadratic is linear iff $B_\varphi \simeq 0$.

**Pf**: To get $(*)$, consider the strongly cocartesian 2-cube

$$\begin{tikzcd} & x \oplus x \arrow[rr] \arrow[dl, "{(1,1)}"'] \arrow[dd] & & x \oplus y \arrow[dl, "{(a',1)}"'] \arrow[dd] \\ x \arrow[rr, crossing over] \arrow[dd] & & y & \\ & z \oplus x \arrow[rr] \arrow[dl, "{(1,b')}"] & & z \oplus y \arrow[dl, "{(a,b)}"] \\ z \arrow[rr] & & w \arrow[from=uu, crossing over] & \end{tikzcd}$$

This gives us

$$\begin{tikzcd} \varphi(w) \arrow[r] \arrow[d] & \varphi(y) \arrow[d] \\ \varphi(z) \arrow[r] & \varphi(x) \end{tikzcd} \overset{(A)}{\Longrightarrow} \begin{tikzcd} \varphi(z \oplus y) \arrow[r] \arrow[d] & \varphi(x \oplus y) \arrow[d] \\ \varphi(z \oplus x) \arrow[r] & \varphi(x \oplus x) \end{tikzcd} \overset{(B)}{\Longrightarrow} \begin{tikzcd} B(z,y) \arrow[r] \arrow[d] & B(x,y) \arrow[d] \\ B(z,x) \arrow[r] & B(x,x) \end{tikzcd}$$

$\varphi(z \oplus y)^{red} = B(z,y)$

WTS (A) & (B) induce equivalences on total fibres.

• For (A), $\varphi$ 2-excisive

$$\begin{tikzcd} F \arrow[r] \arrow[d, equal] & \varphi(w) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \varphi(z) \times_{\varphi(x)} \varphi(y) \arrow[d] \\ F \arrow[r] & \varphi(z \oplus y) \arrow[r] & \varphi(z \oplus x) \times_{\varphi(x \oplus x)} \varphi(x \oplus y) \end{tikzcd}$$

✓

• For (B), taking square of fibres

$$\begin{tikzcd} 0 \arrow[d] & & \\ \mathrm{fib}(\varphi(y) \to \varphi(x)) \arrow[r] \arrow[d] & \varphi(z) \oplus \varphi(y) \arrow[r] \arrow[d] & \varphi(z) \oplus \varphi(x) \arrow[d] \\ \mathrm{fib}(\varphi(y) \to \varphi(x)) \arrow[r] & \varphi(x) \oplus \varphi(y) \arrow[r] & \varphi(x) \oplus \varphi(x) \end{tikzcd}$$

✓ HA.6.1.3.22

• Finally can compute the fibres $B(\mathrm{cofib}(b'), \mathrm{cofib}(a'))$ $B_\varphi$

$$\begin{tikzcd} B(z, \mathrm{cofib}(a')) \arrow[r] \arrow[d] & B(z, y) \arrow[r] \arrow[d] \arrow[r, bend left] & B(z, x) \arrow[d] \\ B(x, \mathrm{cofib}(a')) \arrow[r] & B(x, y) \arrow[r] & B(x, x) \end{tikzcd}$$

✓ $\square$

**Prop** (Characterization of quadratics):
Let $\varphi : \mathcal{C}^{op} \to Sp$ be functor. TFAE:

(1) $\varphi$ is quadratic $\quad$ $\mathrm{cofib}((B^{\Delta}_{\varphi})_{hC_2} \Rightarrow \varphi)$

(2) $B_\varphi$ bilinear & $L_\varphi$ is linear

(3) $B_\varphi$ bilinear & $cL_\varphi$ is linear.

**Pf:** $(2),(3) \Rightarrow (1)$:

HA.6.1.3.5 $\Rightarrow$ $B^\Delta_\varphi$ is 2-excisive reduced
$\Rightarrow$ " quadratic.

$\Rightarrow (B^\Delta_\varphi)_{hC_2}$ is quadratic
quadratic

linear $\Rightarrow$ quadratic

By defn $(B^\Delta_\varphi)_{hC_2} \Rightarrow \varphi \Rightarrow L_\varphi$

$\mathrm{Fun}^{\mathrm{q}}(\mathcal{C})$
" closed under (co)limits.

$(1) \Rightarrow (2),(3)$:

HA.6.1.3.22 $\Rightarrow$ $B_\varphi$ bilinear.

$B_\varphi$ quadratic follows by exactness,

WTS $B_{L_\varphi} \simeq 0$.

$$B_{L_\varphi} = B\left(\mathrm{cofib}\left((B^\Delta_\varphi)_{hC_2} \Rightarrow \varphi\right)\right)$$

$$= \mathrm{cofib}\left(\left(B_{B^\Delta_\varphi}\right)_{hC_2} \Rightarrow B_\varphi\right)$$

$$= \mathrm{cofib}\left(B_\varphi \Rightarrow B_\varphi\right) = 0.$$

$\square$

# 3) Adjunction galore

(1)
$$\begin{tikzcd}
\mathrm{Fun}_*(\mathcal{C}^{op}, Sp) \arrow[r, "\oplus^*"] & \mathrm{Fun}_*(\mathcal{C}^{op} \times \mathcal{C}^{op}, Sp) \arrow[l, bend right, "\Delta^*"'] \arrow[l, bend left, "\Delta^*"] \arrow[r, "(-)^{red}"] & \mathrm{BiFun}(\mathcal{C}) \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Fun}^q(\mathcal{C}) \arrow[u] & \mathrm{Fun}^b(\mathcal{C}) \arrow[u]
\end{tikzcd}$$

(2)
$$\begin{tikzcd}
\mathrm{Fun}^q(\mathcal{C}) \arrow[r, "B_{(-)}"] & \mathrm{Fun}^b(\mathcal{C}) \arrow[l, bend right, "\Delta^*"'] \arrow[l, bend left, "\Delta^*"]
\end{tikzcd}$$

(3)
$$\begin{tikzcd}
\mathrm{Fun}^q(\mathcal{C}) \arrow[r, bend left, "L"] \arrow[r, bend right, "cL"'] & \mathrm{Fun}^{ex}(\mathcal{C}) \arrow[l]
\end{tikzcd}$$

Pf: WTS: If $q \in \mathrm{Fun}^q(\mathcal{C})$, $\xi \in \mathrm{Fun}^{ex}(\mathcal{C})$,

then $\mathrm{Nat}_{ex}(L_q, \xi) \simeq \mathrm{Nat}_q(q, \xi)$

ie WTS $\mathrm{Nat}((B_q^\Delta)_{hC_2}, \xi) \simeq 0$

quad bilinear is examen

$$\mathrm{Nat}((B_q^\Delta)_{hC_2}, \xi) \simeq \mathrm{Nat}(\underbrace{B_q^\Delta}_{\Delta^* B_q}, \xi)^{hC_2} \simeq \mathrm{Nat}(B_q, \cancel{B_{\xi}}^{\,0})^{hC_2}$$

$$\simeq 0$$

$\square$

(4)

$$\begin{tikzcd}
\mathrm{Fun}^q(\mathcal{C}) \arrow[rr, "B_{(-)}"', "\varphi(X \dashv Y)^{red}" description] & & \mathrm{Fun}^s(\mathcal{C}) \arrow[ll, bend right, "\varphi^q_{(-)} = B(Y,X)_{unc}"'] \arrow[ll, bend left, "\varphi^s_{(-)} = B(X,X)^{unc}"]
\end{tikzcd}$$

$\mathrm{Im}(\varphi^q_{(-)}) =$ homogeneous functors

<u>Pf</u>: NTS

(a) $\varphi^q_{(-)}$ is fully faithful

($\checkmark$ since $B_{\varphi^q_\beta} \simeq \beta$)

(b) $\varphi^q_{(-)} \dashv B_\varphi$

(else since $L_{\varphi^q_{(-)}} \simeq 0$)

$\square$

# 4) Quadratic stable recollement

**Def** (Bousfield localisation):

$$\mathcal{C} \underset{}{\overset{L}{\rightleftarrows}} \mathcal{D}$$

**Fact**: Let $\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{E}$ in $\widehat{\mathrm{Cat}}_\infty^{ex}$ s.t. $pi \simeq 0$.

(1) It is a fib seq iff • $i$ is fully faithful
• $\mathrm{Im}\, i \subseteq \ker p$ is an equiv

(2) If $\mathcal{D} \xrightarrow{p} \mathcal{E}$ is Bousfield loc & $\mathrm{Im}\, i \subseteq \ker p$ is an equiv, then it is a cofibre seq.

II.A.1 – II.A.2

**Def**: Let

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook, "i"] & \mathcal{D} \arrow[l, bend right, "\bar{g}"'] \arrow[l, bend left, "g"] \arrow[r, two heads, "p"] & \mathcal{E} \arrow[l, bend right, "\bar{z}"'] \arrow[l, bend left, "z"] \end{tikzcd}$$

in $\widehat{\mathrm{Cat}}_\infty^{ex}$ s.t. $pi \simeq 0$.

Then we say it is a $\boxed{\text{stable recollement}}$ if

(A) it's a fibre seq

(B) $p$ is a Bousfield localisation

(C) $i$ admits a left adj (i.e. $i$ is a Bousfield inclusion)

**Lem** (Characterization of stable recollement): Supp have

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] & \mathcal{D} \arrow[l, bend right, "\bar{g}"'] \arrow[l, bend left, "g"] \arrow[r, "p"] & \mathcal{E} \arrow[l, bend right, "\bar{q}"'] \arrow[l, bend left, "q"] \end{tikzcd}$$

s.t. $pf \simeq 0$.

Then TFAE:

(1) It is a stable recollement (ie $(p, f)$ is a bfs seq).

(2) The square

$$\begin{tikzcd} \mathrm{Id}_{\mathcal{D}} \arrow[r] \arrow[d] & qp \arrow[d] \\ f\bar{g} \arrow[r] & f\bar{g}qp \end{tikzcd}$$

cartesian.

(3) $p, \bar{g}$ are jointly conservative: $(p(d) \simeq 0, \bar{g}(d) \simeq 0) \Rightarrow (d \simeq 0)$.

In this case we have a canonical identification

$$\bar{g}qp \simeq \mathrm{cofib}(\bar{q}p \Rightarrow \mathrm{Id} \Rightarrow qp)$$

**Pf**: $(1) \Rightarrow (3)$: So supp $p(d) \simeq 0$, $\bar{g}(d) \simeq 0$.

Then $(1) \Rightarrow d \simeq f(c)$, some $c \in \mathcal{C}$.

$\Rightarrow 0 \simeq \bar{g}(d) \simeq \bar{g}f(c) \simeq c \Rightarrow f(c) \simeq d \simeq 0$

$(3) \Rightarrow (2)$:

$$\simeq \left[\begin{tikzcd} F \arrow[d, "\alpha"'] \\ f\bar{g}F \end{tikzcd}\right] \begin{tikzcd} {} \arrow[r] & \mathrm{Id}_{\mathcal{D}} \arrow[r] \arrow[d] & qp \arrow[d] \\ {} \arrow[r] & f\bar{g} \arrow[r] & f\bar{g}qp \end{tikzcd}$$

WTS $\alpha$ is an equiv.

$pF \simeq 0$ & $pf \simeq 0 \Rightarrow \underset{0}{\overset{\shortparallel}{pF}} \xrightarrow[\simeq]{p\alpha} \underset{0}{\overset{\shortparallel}{pf\bar{g}F}}$

$$\bar{g}F \xrightarrow{\bar{g}\alpha} \underbrace{\bar{g}f}_{\mathrm{Id}}\bar{g}F$$

□

## Examples:

• (Arithmetic square at $p$)

$$\begin{tikzcd} Sp[1/p] \arrow[r, hook] & Sp \arrow[l, bend right, "(-)[1/p]"'] \arrow[r, "(-)^\wedge_p"] & Sp^\wedge_p \arrow[l, hook, bend left] \end{tikzcd}$$

Will $\boxed{\mathrm{Im}(Sp[1/p] \hookrightarrow Sp) \simeq \ker((-)^\wedge_p).}$

So suppose $X^\wedge_p \simeq 0$. $\Rightarrow$ $X/p \simeq 0$.

$\Rightarrow$ $X \xrightarrow[\simeq]{p} X \longrightarrow \cancel{X/p}$ $\neq 0$

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^\wedge_p \arrow[d] \\ X[1/p] \arrow[r] & X^\wedge_p[1/p] \end{tikzcd}$$

• (Genuine $C_p$-equivariance) $Sp^{C_p} = \mathrm{Fun}^\times(\mathrm{Span}(C_p), Sp)$.

$$\begin{tikzcd} Sp \arrow[r, hook] & Sp^{C_p} \arrow[l, bend right, "\Phi^{C_p}"'] \arrow[r, "ev_e"] & Sp^{BC_p} \arrow[l, hook, bend left] \end{tikzcd}$$

$C_p/C_p$

$C_p/e$

$$X \longmapsto \begin{bmatrix} X \\ 0 \end{bmatrix} \longmapsfrom 0$$

$$\begin{tikzcd} X^{C_p} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Phi^{C_p}X \arrow[d] \\ X^{hC_p} \arrow[r] & X^{tC_p} \end{tikzcd}$$

**Thm** (Quadratic stable recollement): I.1.3.12

(1)
$$\begin{tikzcd} \mathrm{Fun}^{ex}(\mathcal{C}) \arrow[r, hook] & \mathrm{Fun}^q(\mathcal{C}) \arrow[l, bend right, "L"'] \arrow[l, bend left, "cL"] \arrow[r, "B_{(-)}"] & \mathrm{Fun}^s(\mathcal{C}) \arrow[l, bend right, "\text{Ϙ}^q_{(-)}"'] \arrow[l, bend left, "\text{Ϙ}^s_{(-)}"] \end{tikzcd}$$

is a stable recollement.

(2) So
$$\begin{tikzcd} & \text{Ϙ} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & L\text{Ϙ} \arrow[d] \\ \text{Ϙ}^q_{B\text{Ϙ}} \arrow[r, "N"] & \text{Ϙ}^s_{B\text{Ϙ}} \arrow[r] & L\text{Ϙ}^s_{B\text{Ϙ}} \end{tikzcd}$$

, moreover $\left(\text{Ϙ}^s_{B\text{Ϙ}} \to L\text{Ϙ}^s_{B\text{Ϙ}}\right) \simeq \left(B(X,X)^{hC_2} \xrightarrow{\mathrm{can}} B(X,X)^{tC_2}\right)$

obvious (by defn)

non-obvious!

PC: (1) (A) Quadratic functor & exactness.

(B) ✓

(C) ✓

this talk

$$\alpha^{\vee} \varphi^q_{B_\varphi} \xRightarrow{\varepsilon} \varphi \xRightarrow{\eta} \varphi^s_{B_\varphi}$$

(2) $$\left(\varphi^q_{B_\varphi} \underset{N}{\overset{\alpha}{\rightrightarrows}} \varphi^s_{B_\varphi}\right)(X) = \left(B(X,X)_{hC_2} \underset{N}{\overset{\alpha}{\rightrightarrows}} B(X,X)^{hC_2}\right)$$

general $C_2$-norm

Observe: $$B_{(-)} : \mathrm{Nat}(\varphi^q_{B_\varphi}, \varphi^s_{B_\varphi}) \xrightarrow{\simeq} \mathrm{Nat}(B_\varphi, B_\varphi)$$

unwinding adjunctions $\quad \alpha, N \longmapsto B_\alpha \simeq B_N$

- Again unwinding adj, $B_\alpha \simeq \mathrm{Id} : B_\varphi = B_\varphi$ ✓
- OTOH, $B_{(-)}$ preserves (co)limits, so it preserves norm maps.

$$B\left(\varphi^q_{B_\varphi, X} \xrightarrow{N} \varphi^s_{B_\varphi}\right) \simeq \left(B(X,X)\otimes B(X,X)\right)_{hC_2} \xrightarrow{N} \left(B(X,X)\otimes B(X,X)\right)^{hC_2}$$

$$\simeq \left(B(X,X) \xrightarrow{\mathrm{Id}} B(X,X)\right)$$

□

# Schematic summary:

$$
\begin{tikzcd}
 & cL_\varphi(X) \arrow[r, equal] \arrow[d, "\text{colinear}"', "\simeq"] & cL_\varphi(X) \arrow[d] \\
B_\varphi(X,X)_{hC_2} \arrow[r, "\simeq", "\text{homogeneous}"'] \arrow[d, equal] & \varphi(X) \arrow[r, "\simeq", "\text{linear}"'] \arrow[d, "\text{cohomogeneous}"', "\simeq"] \arrow[dr, phantom, "\lrcorner"] & L_\varphi(X) \arrow[d] \\
B_\varphi(X,X)_{hC_2} \arrow[r] & B_\varphi(X,X)^{hC_2} \arrow[r] & B_\varphi(X,X)^{tC_2}
\end{tikzcd}
$$

# 5) Hermite & Poincaré

**Def:** A pair $(\mathcal{C}, \varphi : \mathcal{C}^{op} \to \mathcal{S}p)$ is **hermitian** if $\varphi$ is quadratic.

**Goal:** To define Poincaré categories. Need notion of non-degeneracies in the sense:
sym bilin form $\varphi : M \times M \to R$ inducing iso $M \xrightarrow{\simeq} \mathrm{Hom}_R(M,R)$.

**Cons** (Non-degeneracy & Perfectness as properties):

(1) $\beta \in \mathrm{Fun}^s(\mathcal{C})$, $\quad Y \longmapsto \beta(-,Y)$

$$\begin{tikzcd}
\mathcal{C}^{op} \arrow[r, "\tilde\beta"] \arrow[dr, dashed, "\text{if } \exists"'] & \mathrm{Fun}^{ex}(\mathcal{C}^{op}, \mathcal{S}p) \\
& \mathcal{C} \arrow[u, hook, "j"']
\end{tikzcd}$$

$j$ Yoneda

$\Rightarrow D : \mathcal{C}^{op} \longrightarrow \mathcal{C}$ duality functor.

If this holds, then say that $(\mathcal{C}, \beta)$ is non-degenerate.

(2) If $(\mathcal{C}, \beta)$ was non-degenerate. $\quad D : \mathcal{C}^{op} \longrightarrow \mathcal{C}$, $\quad D^{op} : \mathcal{C} \longrightarrow \mathcal{C}^{op}$

Observe that if $X, Y \in \mathcal{C}$

$$\begin{aligned}
\mathrm{Map}_{\mathcal{C}}(X, DY) \simeq \beta(X,Y) &\simeq \beta(Y,X) \\
&\simeq \mathrm{Map}_{\mathcal{C}}(Y, DX) \\
&\simeq \mathrm{Map}_{\mathcal{C}^{op}}(D^{op}X, Y)
\end{aligned}$$

$\Rightarrow D^{op} \dashv D$

$\Rightarrow \eta : \mathrm{Id} \Rightarrow DD^{op}$

If $\eta$ is an equiv, then say that $(\mathcal{C}, \beta)$ is perfect.

**Def:** A hermitian cat $(\mathcal{C}, Ϙ)$ is Poincaré if $B_Ϙ$ is perfect.

**Rmk:** Hermitian $\longleftrightarrow$ structure

Poincaré $\longleftrightarrow$ property

**Warning:** $\mathrm{Cat}^p \subseteq \mathrm{Cat}^h$ non-full!