I. Modules w/ genuine involution. §3.2.

$$\mathbb{S} \longrightarrow k \longrightarrow A$$
$\mathbb{E}_\infty$ $\mathbb{E}_1$

$$L_\varphi : \mathrm{Perf}(A)^{op} \longrightarrow \mathrm{Sp}$$

$D(A) \simeq \mathrm{Mod}_A(\mathrm{Sp})$ $L_\varphi(X) \simeq \underline{\mathrm{Map}}_A(X, N)$ ← depends on φ / L_φ.

φ Hermitian on $\mathrm{Perf}(A) \simeq D(A)^\omega$.

$$\begin{tikzcd} \varphi(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & L_\varphi(X) \simeq \underline{\mathrm{Map}}_A(X,N) \arrow[d] \\ B_\varphi(X,X)^{hC_2} \arrow[r] & B_\varphi(X,X)^{tC_2} \simeq \underline{\mathrm{Map}}_A(X, M^{tC_2}) \end{tikzcd}$$

$N \xrightarrow{\text{map of } A\text{-mods.}} M^{tC_2}$

$$B_\varphi(X,X)^{hC_2} \simeq \underline{\mathrm{Map}}_{A\otimes A}(X\otimes X, M)^{hC_2}$$

Exact.

$$M \in D(A\otimes A)^{hC_2}$$

Subtlety: M^{tC_2} is an $(A\otimes A)^{tC_2}$-mod

$A \nearrow$ Tate diagonal.

Ex.
$$\mathbb{Z} \longrightarrow (\mathbb{Z}\otimes_{\mathbb{S}}\mathbb{Z})^{tC_2} \longrightarrow \mathbb{Z}^{tC_2}$$
$$\mathbb{S} \longrightarrow (\mathbb{S}\otimes_{\mathbb{S}}\mathbb{S})^{tC_2} \simeq \mathbb{S}^{tC_2} \simeq \mathbb{S}_2^\wedge$$
$$\mathbb{F}_2 \longrightarrow (\mathbb{F}_2\otimes_{\mathbb{S}}\mathbb{F}_2)^{tC_2} \longrightarrow \mathbb{F}_2^{tC_2}$$
"$\prod_{i\geq 0} Sq^i$" $\qquad \mathbb{F}_2^{tC_2} \simeq \prod_{i\in\mathbb{Z}} \mathbb{F}_2[i]$

On π_0, R commutative discrete

$$\begin{tikzcd} R \arrow[r] \arrow[d, "\mathrm{can}"'] & \pi_0 R^{tC_2} \simeq R/2 \\ R/2 \arrow[ur, "\varphi"'] & \end{tikzcd}$$

Def. Module w/ genuine involution of A is a triple $(N \xrightarrow{\alpha} M^{tC_2})$, $N \in D(A)$, $M \in D(A\otimes A)^{hC_2}$, α is an A-module map.

Constr.
$$\begin{tikzcd} \varphi_M^\alpha(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \underline{\mathrm{Map}}_A(X,N) \arrow[d, "\alpha"] \\ \varphi_M^s(X) \arrow[r] & \underline{\mathrm{Map}}_A(X, M^{tC_2}) \end{tikzcd}$$
.

$$\varphi^s_M(X) := \underline{\mathrm{Map}}_{A\otimes A}(X\otimes X, M)^{hC_2} \quad \Big| \quad (M^{tC_2} \xrightarrow{id} M^{tC_2})$$

$$\varphi^q_M(-) := \big(\underline{\qquad\qquad} \big)_{hC_2} \quad \Big| \quad (0 \longrightarrow M^{tC_2})$$

Thm. $\mathrm{Fun}^q(\mathrm{Perf}(A)) \simeq \{\text{modules w/ genuine involution}\}$.

Rem. φ^α_M is nondegenerate $\iff$ M is compact as an A-module along either $A \to A\otimes A$.

φ^α_M is Poincaré if $A \simeq \underline{\mathrm{Map}}_A(M, M)$.

Ex. $M \in D(A\otimes A)^{hC_2}$, A connective.

$$\tau_{\geq n} M^{tC_2} \xrightarrow{\alpha} M^{tC_2} \longleftarrow \text{Module w/ gen. inv.}$$

$$\varphi^q_M \cdots \to \varphi^{\geq n}_M \to \varphi^{\geq n-1}_M \to \cdots \varphi^s_M$$

II. Periodicity. §3.4.

$$\varphi^{[n]} := \Sigma^n \circ \varphi : \mathcal{C}^{op} \to Sp \xrightarrow{\Sigma^n} Sp \qquad (n \in \mathbb{Z}).$$

"n-shifted"

Ex. $(\varphi^\alpha_M)^{[n]} \simeq \varphi^{\Sigma^n \alpha}_{\Sigma^n M}$ $\qquad \Sigma^n N \xrightarrow{\Sigma^n \alpha} (\Sigma^n M)^{tC_2}$.

Warning. $\varphi \circ \Sigma^n : \mathcal{C}^{op} \xrightarrow{\Sigma} \mathcal{C}^{op} \xrightarrow{\varphi} Sp$

more difficult to understand exactly because φ is not exact.

$$(\mathcal{C}, \varphi\circ\Sigma^n) \xrightarrow[\Sigma^n]{\simeq} (\mathcal{C}, \varphi)$$

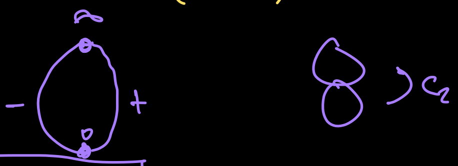
Wellspring of Periodicity.

$$(\varphi^\alpha_M)^{[m+n]} \circ \Sigma^n \simeq \varphi^{\Sigma^m \alpha}_{\Sigma^{m-n\sigma} M}.$$

S^σ = 1 pt. cpt of sign rep on $\mathbb{R}$.

Lemma. $S^0 \to S^\sigma$ induces $X^{tC_2} \to (\Sigma^\sigma X)^{tC_2}$ for $X \in Sp^{BC_2}$.

$S^0 \subseteq$



$$X \to \Sigma^\sigma X \to C \simeq \boxed{\Sigma_+ C_2 \wedge X}$$

induced $\Rightarrow$ $(-)^{tC_2}$ vanishes.

Proof of ..., $\Sigma^m N \xrightarrow{\Sigma^m \alpha} (\Sigma^m M)^{tC_2} \simeq (\Sigma^{m+n\sigma} M)^{tC_2}$.

Assume $m=0$.

$$\begin{array}{ccc}
\Sigma^n \varphi(\Sigma^n X) & \longrightarrow & \Sigma^n \underline{M}\cdot p(\Sigma^n X, N) \simeq \underline{M}\cdot p(X,N) \\
\downarrow & \lrcorner & \downarrow \\
\Sigma^n \underline{M}\circ p_{A\otimes A}(\Sigma^n X \otimes \Sigma^n X, M)^{hC_2} & \longrightarrow & \Sigma^n \underline{M}\cdot p_{A\otimes A}(\Sigma^n X \otimes \Sigma^n X, M)^{tC_2} \\
& & \wr \\
& & \Sigma^n \underline{M}\cdot p_{A\otimes A}(\Sigma^{n+n\sigma} X \otimes X, M)^{tC_2} \\
& & \wr \\
& & \underline{M}\cdot p_{A\otimes A}(X \otimes X, \Sigma^{-n\sigma} M)^{tC_2}
\end{array}$$

Def. k $\mathbb{E}_\infty$-ring.

k admits a symmetry if

$$k \simeq \Sigma^{2-2\sigma} k$$

or $\Sigma^2 k \simeq \Sigma^{2\sigma} k$.

Symmetry $\Rightarrow$ k^{tC_2} is 2-periodic.

Exercise!

Ex. MU, ku, discrete comm. rings admit symmetries.
But, $\mathbb{S}$ does not admit a symmetry.

If k has a symmetry, let $-k = \Sigma^{1-\sigma} k$.
Similarly $-M \simeq \Sigma^{1-\sigma} M$.

Cor. $(\mathrm{Perf}(A), (\varphi^\alpha_M)^{[2n]}) \underset{\Omega^n}{\simeq} (\mathrm{Perf}(A), \Sigma^{2n}\circ\varphi\circ\Sigma^n)$ (m = n in addl pr.)

$\wr$

$(\mathrm{Perf}(A), \varphi^{\Sigma^n \alpha}_{\Sigma^{n-n\sigma} M})$

Periodicity II. k $\mathbb{E}_\infty$-ring with a symmetry, A an $\mathbb{E}_1$-k-algebra,

$M \in D(A \otimes_k A)^{hC_2}$. $\underline{N \xrightarrow{\alpha} M^{tC_2}}$.

(A) $\left(\mathrm{Perf}(A), (\Omega^s_M)^{[4]}\right) \underset{\Omega}{\simeq} \left(\mathrm{Perf}(A), (\Omega^s_{-M})^{[2]}\right) \underset{\Omega}{\simeq} \left(\mathrm{Perf}(A), \Omega^s_M\right)$.

Same for Ω^q_M ....

(B) $\left(\mathrm{Perf}(A), (\Omega^q_M)^{[4]}\right) \underset{\Omega^2}{\simeq} \left(\mathrm{Perf}(A), \Omega^{\Omega^2\alpha}_M\right)$.

(C) $\left(\mathrm{Perf}(A), (\Omega^{\geq m}_M)^{[4]}\right) \underset{\Omega}{\simeq} \left(\mathrm{Perf}(A), (\Omega^{\geq m+1}_{-M})^{[2]}\right) \underset{\Omega}{\simeq} \left(\mathrm{Perf}(A), \Omega^{\geq m+2}_M\right)$

Gossip.

$$[\omega \to x] \longmapsto \mathrm{Fib}(\omega \to x)$$

$$\mathrm{Pn}(\mathrm{Met}(\mathcal{C}, \Omega)) \simeq \mathrm{Fm}(\mathcal{C}, \Omega^{[-1]})$$

Poincaré obj. of $\mathcal{C}$ w/ Lagrangian

$$\mathrm{Pn}(\mathrm{Ac}(\mathcal{C}, \Omega^{[-1]}))$$

$$C^\cdot(\mathrm{Th}(\nu), \mathbb{Z}) \to C^\cdot(M^n, \mathbb{Z}) \to C^\cdot(U, \mathbb{Z})$$

Null bordisms.

$(-2n)$-shifted

$N^n \subseteq M^{2n}$

collar U open

$\underline{\text{Lefschetz}}$

III. Genuine / derived Poincaré cat etc. §4.2.

$\underline{\text{Def}}$. A functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ between additive ∞-cats is $\underline{\text{additively quadratic}}$ if $B_F(-,-)$ is biadditive and F is reduced.

Prop. A an associative ring,

$$\mathrm{Fun}^q(\mathrm{Perf}(A)^{op}, Sp) \xrightarrow{\sim} \mathrm{Fun}^{q}(\mathrm{Proj}^{\omega}(A)^{op}, Sp).$$

Ex. $\varphi^a : \mathrm{Proj}^{\omega}(A)^{op} \to Sp$, $\quad X \in \mathrm{Perf}(A)_{\geq 0}$

$$X \cong |P_\bullet|$$

$$\varphi(X) \simeq \mathrm{Tot}(\varphi(P_\bullet)).$$

Ex. $M \in \mathrm{Mod}^{hC_2}_{A \otimes_{\mathbb{Z}} A}$.

$$\varphi^{gq}_M(P) = \mathrm{Hom}_{A\otimes A}(P\otimes P, M)_{C_2} \qquad \text{genuine quadratic}$$

$$\varphi^{gs}(P) = (\quad\text{—}\quad)^{C_2} \qquad \text{genuine symmetric}$$

$$\varphi^{ge}(P) = \mathrm{Im}(\varphi^{gq}_M(P) \to \varphi^{gs}_M(P)).$$

Rem. $\varphi^{gs}_M(P) \simeq \tau_{\geq 0}\, \varphi^{s}_M(P)$.

Prop.

$$\begin{tikzcd}
\varphi^{gq}_M \arrow[r] & \varphi^{ge}_M \arrow[r] & \varphi^{gs}_M \\
\varphi^{\geq 2}_M \arrow[u, "\cong"] \arrow[r] & \varphi^{\geq 1}_M \arrow[u, "\cong"] \arrow[r] & \varphi^{\geq 0}_M \arrow[u, "\cong"]
\end{tikzcd}$$

$$\tau_{\geq 2} M^{tC_2} \to \boxed{M^{tC_2}}$$

Proof.

$$\begin{tikzcd}
\varphi^{gs}_M(P) \arrow[r] \arrow[d, "\tau_{\geq 0}"'] & L_{\varphi^{gs}_M}(P) \simeq \underline{\mathrm{Map}}_A(P, \tau_{\geq 0} M^{tC_2}). \arrow[d] \\
{} & {}
\end{tikzcd}$$

$$\varphi^s_M(P) \longrightarrow \underline{\mathrm{Map}}_A(P, M^{tC_2})$$

Periodicity II. (c) $(\mathrm{Perf}(A), (\varphi^{\geq m}_M)^{[4]}) \underset{\Omega}{\simeq} (\mathrm{Perf}(A), (\varphi^{\geq m+1}_{-M})^{[2]}) \underset{\Omega}{\simeq} (\mathrm{Perf}(A), \varphi^{\geq m+2}_M)$

A ass. ring.

$M \in \mathrm{Mod}^{hC_2}_{A \otimes_{\mathbb{Z}} A}$

$$\left(\mathrm{Perf}(A), (\varphi^{gs}_M)^{[4]}\right) \underset{\Omega}{\simeq} \left(\mathrm{Perf}(A), (\varphi^{gs}_{-M})^{[2]}\right) \underset{\Omega}{\simeq} \left(\mathrm{Perf}(A), \varphi^{gs}_M\right).$$

IV. Symmetric monoidal structures. §5.4.

Thm. A be a $\mathbb{E}_\infty$-ring. Symmetric monoidal Hermitian str. on $\mathrm{Perf}(A)$

$$C \in \mathrm{CAlg}_A, \quad B \in \mathrm{CAlg}^{hC_2}_{A \otimes A},$$

$$C \to B^{tC_2}$$

$\mathbb{E}_\infty$-A-algebra.

$\mathbb{E}_\infty$-A-alg. w/ genuine inv.

Such a structure is Poincaré if $A \simeq B$.

Ex. Fix $A \in \mathrm{CAlg}^{hC_2}$.

$$\begin{array}{lll} \varphi^s & A^{tC_2} \overset{\mathrm{id}}{=\!=} A^{tC_2} & \text{Final.} \\ \uparrow & & \\ \varphi^{gs} & \tau_{\geq 0} A^{tC_2} \longrightarrow A^{tC_2} & A \text{ is connective} \\ \uparrow & & \\ \varphi^{t} & A \xrightarrow[\text{diagonal}]{\text{Tate}} A^{tC_2} & \text{Initial.} \end{array}$$

Ex. A commutative ring w/ trivial C_2-action. Maybe 2-torsion free.

P f.p. proj. A-module.

$$\Omega^\infty \varphi^s(P) \simeq \Omega^\infty \varphi^{gs}(P) \quad \text{symm. bilin. forms on } P.$$

$$\begin{tikzcd}
\varphi^q(P) \arrow[r] \arrow[d, "\wr"] & \varphi^t(P) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \underline{\mathrm{Map}}_A(P,A) \arrow[d] \\
\varphi^q(P) \arrow[r] & \varphi^s(P) \arrow[r] & \underline{\mathrm{Map}}_A(P,A^{tC_2})
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[d] \\
A^{tC_2}
\end{tikzcd}$$

$b(-,-)$ $\qquad F(x) = b(x,x) \bmod 2$ in $A/2 = \pi_0 A^{tC_2}$.

$b(x,x) \in 2A$

$F(x+y) = F(x) + F(y)$

$F(ax) = a^2 F(x)$.

b lifts to a quadratic form $\iff F = 0$.

$F : P \to \cancel{\varphi^*} A/2$ lin

$\varphi^* P \to A/2$.

Diagonal entries of B.

$B^t = B$

A Tate form on P is a pair (b, F) of a symmetric bilin form b with a lift

$$\begin{tikzcd}
 & A \arrow[d, "\text{Tate diagonal}"] \\
P \arrow[r, "f_b"'] \arrow[ur, dashed, "F"] & \varphi_* A/2 .
\end{tikzcd}$$

Ex. $\mathbb{Z}[x]$. $\quad B = (x)$

Does not lift.