9/30/2021

**Disclaimers.**

- All categories are $\infty$-categories. Most are stable.
- Hom, $\otimes$, etc. are derived. Write Mod(R) for D(R).
- Any mistakes are my own
- manifolds/as topological space vs. "space" for homotopy type.

# §.0. Recollection $(+\varepsilon)$

analogy for functions $f: \mathbb{R}^k \to \mathbb{R}$.

$\mathcal{C}$ (small) stable $\infty$-category $\rightsquigarrow$ categories of functors.

(assume everything in sight is reduced). $f(0)=0$.

- exact functors $\mathrm{Fun}^{ex}(\mathcal{C}) = \mathrm{Fun}^{ex}(\mathcal{C}^{op}, Sp)$ $\quad f(x) = a \cdot x$.
- bilinear bifunctors. $\quad b(x,y) = x^T A y$

  $$\mathrm{Fun}^b(\mathcal{C})$$

- symmetric bilinear bifunctors. $\quad$ s.t. $b(x,y) = b(y,x)$.

  $$\mathrm{Fun}^b(\mathcal{C})^{hC_2}$$

- quadratic functors: $\quad f(x) = ax^2 + bx$

  $$\mathrm{Fun}^q(\mathcal{C}) \subset \mathrm{Fun}(\mathcal{C}^{op}, Sp).$$

"rattachement".

**Thm** (Quadratic stable recollement)

$$\begin{tikzcd}
\mathrm{Fun}^{ex}(\mathcal{C}) \arrow[r] & \mathrm{Fun}^q(\mathcal{C}) \arrow[l, bend right, "L_{(-)}"'] \arrow[l, bend left] \arrow[r, "B_{(-)}", "\perp"'] & \mathrm{Fun}^s(\mathcal{C}) \arrow[l, bend right, "Ϙ^q_{(-)}"'] \arrow[l, bend left, "\perp", "Ϙ^s_{(-)}"']
\end{tikzcd}$$

(sheaves on an) "open subscheme" $\qquad$ Analogy! (sheaves on a) "closed subscheme"

is a stable recollement.

For any $Ϙ \in \mathrm{Fun}^q(\mathcal{C})$, have a pullback square.

$$\begin{tikzcd}
Ϙ \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & L_{Ϙ} \arrow[d] \\
Ϙ^s_{B_Ϙ} \arrow[r] & L_{Ϙ^s_{B_Ϙ}}
\end{tikzcd}$$

$\leftarrow$ modify the linear part however you want,

**Def** A bilinear functor

$B: \mathcal{C}^{op} \times \mathcal{C}^{op} \longrightarrow Sp$

is <u>non degenerate</u> if

$$\begin{tikzcd}
y \arrow[r, mapsto] & B(-,y). \\
\mathcal{C}^{op} \arrow[r] \arrow[dr, dashed, "\exists D_B"'] & \mathrm{Fun}(\mathcal{C}^{op}, Sp) \\
& \mathcal{C} \arrow[u, "\text{yoneda}"'] \arrow[ul, phantom, "\Uparrow"]
\end{tikzcd}$$

i.e. $B(x,y) \simeq \hom_{\mathcal{C}}(x, D_B y)$.

A quadratic functor $Ϙ$ is nondegenerate if $B_Ϙ$ is.

**Note** If $B$ is symmetric bilinear, then

$$\hom_{\mathcal{C}}(x, D_B y) \simeq B(x,y) \simeq B(y,x) \simeq \hom_{\mathcal{C}}(y, D_B x)$$
$$\simeq \hom_{\mathcal{C}^{op}}(D_B^{op} x, y)$$

i.e. $\quad D_B^{op} \dashv D_B$

**Def** A symmetric bilinear functor $B \in \mathrm{Fun}^s(\mathcal{C})$ is perfect if it is nondegenerate & unit ev : $\mathbb{1} \xrightarrow{\sim} D_B D_B^{op}$

A quadratic $Ϙ \in \mathrm{Fun}^q(\mathcal{C})$ is perfect if $B_Ϙ$ is.

**Lemma** Let $(\mathcal{C}, Ϙ)$, $(\mathcal{D}, \Phi)$ non-degenerate hermitian $\infty$-categories w/ assoc. dualities $D_Ϙ$, $D_\Phi$, & let $f, g : \mathcal{C} \to \mathcal{D}$ exact functors. Then

$$\mathrm{nat}(B_Ϙ, (f\times g)^* B_\Phi) \simeq \mathrm{nat}(f \circ D_Ϙ, D_\Phi \circ g^{op})$$

Spaces of such diagrams are equivalent

$$\begin{tikzcd}
\mathcal{C}^{op}\times\mathcal{C}^{op} \arrow[rr, "f\times g"] \arrow[dr, "B_Ϙ"'] & & \mathcal{D}^{op}\times\mathcal{D}^{op} \arrow[dl, "B_\Phi"] & \longleftrightarrow & \mathcal{C}^{op} \arrow[r, "g^{op}"] \arrow[d, "D_Ϙ"'] & \mathcal{D}^{op} \arrow[d, "D_\Phi"] \\
& Sp \arrow[u, Rightarrow, "\eta", shorten=2mm] & & & \mathcal{C} \arrow[r, "f"'] \arrow[ur, Rightarrow, "\tau", shorten=5mm] & \mathcal{D}
\end{tikzcd}$$

**Recall** A hermitian functor $(\mathcal{C}, Ϙ) \xrightarrow{(f,\eta)} (\mathcal{D}, \Phi)$ is the data of:

- an exact functor $f : \mathcal{C} \to \mathcal{D}$.
- a natural transformation

$$\begin{tikzcd}
\mathcal{C}^{op} \arrow[rr, "f^{op}"] \arrow[dr, "Ϙ"'] & & \mathcal{D}^{op} \arrow[dl, "\Phi"] \\
& Sp \arrow[u, Rightarrow, "\eta", shorten=2mm] &
\end{tikzcd}$$

**Def** A hermitian functor $(f,\eta) : (\mathcal{C}, Ϙ) \to (\mathcal{D}, \Phi)$ is duality preserving if $\tau : f \circ D_Ϙ \xrightarrow{\sim} D_\Phi \circ f^{op}$ is an equivalence.

## §1. Hermite & Poincaré, ctd.

**Q** what is the "moduli of hermitian forms" on a cat $\mathcal{C}$?
quadratic

$$\begin{tikzcd}
 & \widetilde{\mathrm{CAT}}_\infty \arrow[d] \\
(\mathrm{Cat}^{ex}_\infty)^{op} \arrow[r, "\mathrm{Fun}^{Ϙ}(-)"] & \mathrm{CAT}_\infty \\
\mathcal{C} \arrow[r, mapsto] & \mathrm{Fun}^{Ϙ}(\mathcal{C})
\end{tikzcd}$$

← objects are $(\mathcal{C}, \ni x)$.
& functors $(\mathcal{C},x)(\mathcal{D},y)$
are $f: \mathcal{C} \to \mathcal{D}$ & $f(x)=y$.

an object here + a "point" here

**Def** The $\infty$-category of Hermitian $\infty$-cats $\underline{\mathrm{Cat}}^h_\infty$ is the Grothendieck construction applied to

**Aside** What is this? gives an equivalence.

"category fiber bundles w/ 'connection' over $\mathcal{C}$" $\simeq$ "classifying functors. $\mathcal{C} \to [\ -\ ]$"

Recall: $X$ "nice" space
$\left\{ \begin{matrix} \mathbb{C}/\mathbb{R}\text{-vector} \\ \text{bundles on } X \end{matrix} \right\} \simeq \left\{ \begin{matrix} \text{maps} \\ X \to BU, BO \end{matrix} \right\}$.

The $\infty$category $\underline{\mathrm{Cat}}^p_\infty$ of Poincaré $\infty$categories is the (non-full!) subcategory of $\underline{\mathrm{Cat}}^h_\infty$ w/

- objects: $(\mathcal{C}, Ϙ: \mathcal{C}^{op} \to Sp)$ s.t. $Ϙ$ perfect
- morphisms: $(f,\eta): (\mathcal{C}, Ϙ) \to (\mathcal{D}, \Phi)$ duality preserving
  i.e. $\tau: f \circ D_{Ϙ} \xrightarrow{\sim} D_{\Phi} \circ f^{op}$.

"self-adjoint | Hermitian"

## § 2. Examples

**Ex** $R$ commutative (discrete) ring.

Let $\mathcal{C} = \mathrm{Perf}(R)$. $\rightsquigarrow$ consider the symmetric bilinear functor $B_R \in \mathrm{Fun}^s(\mathcal{C})$.

$$B_R(X,Y) = \mathrm{Hom}_R(X \otimes_R Y, R[n]).$$

**Claim:** $B_R$ is perfect w/ duality. $D_{B_R}(Y) = \underset{\sim}{\hom}_R(Y, R[n])$ internal hom!

Can associate

symmetric: & quadratic

$$\text{Ϙ}^{s[n]}_R(X) := B_R(X,X)^{hC_2} \qquad \text{Ϙ}^{q[n]}_R(X) = B_R(X,X)_{hC_2}.$$

**Idea:** a point $\beta \in \Omega^\infty \text{Ϙ}^{s[n]}_R(X) \longleftrightarrow$ symmetric bilinear $R$-form on $X$

**Ex** n-fold shifts.

**Ex** ($E_8$-lattice). Let $\Gamma_8 = \{(x_i) \in \mathbb{Z}^8 \cup (\mathbb{Z}+\frac{1}{2})^8 : \sum x_i \equiv 0 \bmod 8\} \subseteq \mathbb{R}^8$. w/ quadratic form the restriction of the norm (length$^2$) on $\mathbb{R}^8$

**Fact** $(\Gamma_8, \|\cdot\|^2)$ is unimodular.

(Freedman) $\exists$ topological 4-mfld w/ intersection form $\cong$ $E_8$-lattice $\Rightarrow$ does not have smooth structure!

**Ex** Same $\mathcal{C}$, $B_R$ (as a functor!) but w/ minus $C_2$-structure/action, i.e. the map

$$\begin{tikzcd}
B_{-R}(X,Y) \arrow[d, phantom, "\cong" rotate=90] & B_{-R}(Y,X) \arrow[d, phantom, "\cong" rotate=90] \\
\hom_{\mathcal{C}}(X \otimes_R Y, R) \arrow[r, "\sim", "\text{minus that of } B_R"'] & \hom_{\mathcal{C}}(Y \otimes_R X, R)
\end{tikzcd}$$

$$\Rightarrow \text{Ϙ}^s_{-R}(X) = B_{-R}(X,X)^{hC_2} \qquad \text{Ϙ}^q_R(X) = B_{-R}(X,X)_{hC_2}.$$

$$\Omega^\infty(\overset{\downarrow}{-}) = \left\{ \begin{matrix} \text{antisymmetric} \\ \text{forms on } X \end{matrix} \right\}$$

**Ex** "globalize" $X$ scheme ⇝ $\mathcal{C} := \mathrm{Perf}(X)$. — perfect complexes of quasicoherent sheaves.

$\mathcal{L}$ = line bundle on $X$, ⇝ get bilinear form on $\mathcal{C}$:

$$B_{\mathcal{L}}(\mathcal{F},\mathcal{G}) = \mathrm{Hom}_X(\mathcal{F}\otimes\mathcal{G},\mathcal{L}).$$

- perfect w/ duality:

$$D_{\mathcal{L}}(\mathcal{F}) = \hom_X(\mathcal{F},\mathcal{L}).$$ ← internal hom!

- assoc. symmetric & quadratic

$$Ϙ^s_{\mathcal{L}}(\mathcal{F}) = B_{\mathcal{L}}(\mathcal{F},\mathcal{F})^{hC_2} \qquad Ϙ^q_{\mathcal{L}}(\mathcal{F}) = B_{\mathcal{L}}(\mathcal{F},\mathcal{F})_{hC_2}.$$

Poincaré structures.

**Ex** Let $\mathrm{Sp}^f$ = category of finite spectra

Define a hermitian structure $Ϙ^u$ on $\mathrm{Sp}^f$ via pullback:

$$\begin{tikzcd}
Ϙ^u(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & D(X) = \mathrm{Hom}_{\mathrm{Sp}}(X,\mathbb{S}^0) \arrow[dr, "\text{Tate diagonal}"] & \\
D(X\otimes X)^{hC_2} \arrow[r] & D(X\otimes X)^{tC_2} \arrow[r, phantom, "\simeq"] & (DX\otimes DX)^{tC_2}
\end{tikzcd}$$

Spanier-Whitehead dual.

both sides are exact — suffices to check equivalence on $\mathbb{S}^0$.

**Facts**: ▸ This is the universal Poincaré $\infty$-category. (see §3)

▸ unit of a symmetric monoidal structure on $\mathrm{Cat}^{\mathrm{p}}_{\infty}$

Modules w/ involution.

Let $k$ an $\mathbb{E}_\infty$-ring spectrum (base, e.g. $k = \mathbb{S}$ or $\mathbb{Z}$).

& $A$ an $\mathbb{E}_1$-$k$-algebra.

<u>Def</u> A <u>module w/ involution</u> over $A$ is an object $M \in (\mathrm{Mod}_{A\otimes_k A})^{hC_2}$

where $C_2$ acts on $\mathrm{Mod}_{A\otimes_k A}$ via flip action on $A \otimes_k A$.

i.e. a spectrum $M$ w/ $C_2$ action & $A\otimes_k A$-module structure

s.t. the involution of $J$ is linear over flip on $A \otimes_k A$.

<u>Idea</u>: 2 actions of $A$ on $M \in \mathrm{Mod}_k$ $\underset{\text{1st actn.}}{\rightsquigarrow}$ $M \in \mathrm{Mod}_A$.

$\underset{\text{2nd actn.}}{\rightsquigarrow}$ $A \to \mathrm{Hom}_A(M, M)$ action via $A$-module maps.

Equivalence between 2 module structures $\rightsquigarrow$ doesn't matter which is considered first.

Given such an $M$, get a <u>symmetric</u> bilinear functor.

$$B_M : \mathrm{Mod}_A \times \mathrm{Mod}_A \longrightarrow \mathrm{Sp}.$$

$$X, Y \longmapsto \mathrm{Hom}_{A\otimes_k A}(X \otimes_k Y, M).$$

<u>Claim</u> $B_M$ is non-degenerate $\iff$ $M \in \mathrm{Mod}_A^\omega$ cpt as an $A$-module.

$B_M$ is perfect $\iff$ holds <u>and</u> is an equivalence.

$\rightsquigarrow$ assoc. quadratic & symmetric.

$$\text{Ϙ}^q_M(X) := \hom_{A\otimes_k A}(X \otimes_k X, M)_{hC_2} \qquad \text{Ϙ}^s_M(X) := \mathrm{Hom}_{A\otimes_k A}(X\otimes_k X, M)^{hC_2}.$$

<u>Ex</u> If $A$ $\mathbb{E}_1$-algebra w/ anti-involution, i.e. ring map $A \xrightarrow{\overline{(\cdot)}} A^{op}$

view $A$ as an $A \otimes_k A$-module via

$$\begin{array}{c} A \otimes_k A \\ \wr\!\wr \\ A \otimes_k A^{op}. \end{array}$$

i.e. $(a \otimes b) \cdot x = a \cdot x \, \overline{b}$.

**Ex** $\overset{\mathbb{E}_1}{G}$ group. $\chi : G \to \overset{\mathrm{Aut}(\mathbb{S})}{\{\pm 1\} = \mathbb{Z}/2}$ homomorphism.

$\rightsquigarrow$ get involution on group algebra:

$$\mathbb{Z}[G] \longrightarrow \mathbb{Z}[G]^{op}.$$

$$g \longmapsto \chi(g)\cdot g^{-1}$$

$\Sigma^\infty_+$ or $\mathbb{S}[G]$.

Take $G = \pi_1(X,x)$ Poincaré cplx.

That was just the bilinear part.

**Def** An <u>module w/ genuine involution</u> over $A$ is a triple.

- invertible module w/ involution $M$ over $A$.
- an $A$-module $N$
- an $A$-linear map $\alpha : N \to M^{tC_2}$

$(A\otimes_k A)^{tC_2}$-module $\Downarrow$ $A$-module via Tate diagonal $A \to (A\otimes_k A)^{tC_2}$.

Induces a quadratic functor on $\mathrm{Perf}(A)$:

$$\begin{tikzcd}
Ϙ^\alpha_M(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \hom_A(X,N) \arrow[d] \\
\hom_{A\otimes_k A}(X\otimes X, M)^{hC_2} \arrow[dr] & \hom_A(X, M^{tC_2}) \arrow[d, phantom, "\wr"] \\
 & \hom_{A\otimes_k A}(X\otimes X, M)^{tC_2}
\end{tikzcd}$$

exact as functor of $X$.

**Note** $N = 0 \rightsquigarrow Ϙ^q_M$ quadratic

$\vdots$

$N = \tau_{\geq m} M^{tC_2} \rightsquigarrow Ϙ^{\geq m}_M$ something interpolating between $\updownarrow$

$\vdots$

$N = M^{tC_2} \rightsquigarrow Ϙ^s_M$ symmetric.

## §3. Hermitian & Poincaré objects.

**Q** Given a hermitian ∞-cat $(\mathcal{C}, Ϙ : \mathcal{C}^{op} \to Sp)$, & $x \in \mathcal{C}$, what is the "moduli of hermitian forms on $x$"?

**Def** The ∞-category of hermitian objects in $\mathcal{C}$ $He(\mathcal{C}, Ϙ)$.

$$\begin{tikzcd}
\mathcal{C}^{op} \arrow[r, "Ϙ"] & \underline{Sp} \arrow[r, "\Omega^\infty"] & Spc \\
x \arrow[r, mapsto] & Ϙ(x) \arrow[r, mapsto] & \Omega^\infty Ϙ(x)
\end{tikzcd}$$

point in here!

is the Grothendieck construction on the functor $\Omega^\infty Ϙ$

ws gives a right fibration $He(\mathcal{C}) \to \mathcal{C}^{op}$.

Let $Fm(\mathcal{C}, Ϙ) \subset He(\mathcal{C}, Ϙ)$ maximal sub ∞-groupoid.

**Lemma.** The assignment

$$\begin{tikzcd}
\underline{Cat}^h_\infty \arrow[r] & \underline{Cat}_\infty \\
(\mathcal{C}, Ϙ) \arrow[r, mapsto] \arrow[dr] & He(\mathcal{C}, Ϙ) \arrow[d, Rightarrow] \\
 & \mathcal{C}
\end{tikzcd}$$

defines a functor, w/ natural transformation.

**Rmk** (universality) $Map_{\underline{Cat}^h}((Sp^{fin}, Ϙ^u), (\mathcal{C}, Ϙ)) \simeq Fm(\mathcal{C}, Ϙ)$.

**Q** Nondegeneracy ("unimodularity") condition?

**A** Assume $(\mathcal{C}, Ϙ)$ nondegenerate. Then for $(x, q \in \Omega^\infty Ϙ(x)) \in He(\mathcal{C}, Ϙ)$.

$$\begin{tikzcd}
\Omega^\infty Ϙ(x) \arrow[r] & \Omega^\infty B(x,x) \simeq \Omega^\infty \hom_{\mathcal{C}}(x, D_Ϙ x). \\
q \arrow[r, mapsto] & q\#.
\end{tikzcd}$$

**Def** Say a hermitian object $(x,q) \in He(\mathcal{C},\Qoppa)$ is Poincaré if $q_\#: x \to D_\Qoppa x$ is an equivalence.

Write $Pn(\mathcal{C},\Qoppa) \subseteq Fm(\mathcal{C},\Qoppa)$ for full subgroupoid on Poincaré objects.

⚠ Not the sub ∞category of $He(\mathcal{C},\Qoppa)$.

**Lemma** $f: (\mathcal{C},\Qoppa) \to (\mathcal{D},\Phi)$ nondegenerate hermitian functor of nondegenerate hermitian ∞cats. Then

$$\begin{tikzcd} f_*: Fm(\mathcal{C},\Qoppa) \arrow[r] & Fm(\mathcal{D},\Phi) \\ Pn(\mathcal{C},\Qoppa) \arrow[u, phantom, "\cup"] \arrow[r] & Pn(\mathcal{D},\Phi) \arrow[u, phantom, "\cup"] \end{tikzcd}$$

restricts to

In particular, get a functor $Pn : \underline{Cat}^p_\infty \to \underline{Spc}$. highlight.

?? **Pf** The natural transformation $\eta: \Qoppa \to f^*\Phi$ determines

$$\begin{tikzcd} \Omega^\infty \Qoppa(x) \arrow[rr] \arrow[d] & & \Omega^\infty \Phi(f(x)) \arrow[d] \\ \Omega^\infty B(x,x) \arrow[rr] \arrow[d, "\wr"] & & \Omega^\infty B_\Phi(f(x), f(x)) \arrow[d] \\ hom_C(x, D_\Qoppa x) \arrow[r, "f"] & hom_D(fx, f D_\Qoppa x) \arrow[r, "\sim"] & hom_D(fx, D_\Phi f^{op} x) \end{tikzcd}$$

post compose w/ $\tau: f D_\Qoppa \Rightarrow D_\Phi f^{op}$

equivalence if $(f,\eta)$ duality preserving. □

**Ex.** M cpct oriented topological n-manifold w/ fundamental class $[M] \in H_n(M, \partial M; \mathbb{Z})$

⇝ get (−n)-shifted hermitian form:

$$q^M \in \mathrm{Map}_{\mathbb{Z}}\left(C^*(M,\partial M) \otimes_{\mathbb{Z}} C^*(M,\partial M), \mathbb{Z}[-n]\right)^{hC_2}$$

$$\simeq \Omega^\infty \mathcal{Q}^{s[-n]}_{\mathbb{Z}}\left(C^*(M,\partial M)\right).$$

$$(\varphi, \psi) \longmapsto (\varphi \cup \psi)[M].$$

maps in $H\mathbb{Z}$-module spectra.

chains.

homological grading conventions.

Alternatively: as intersection form on $C_*(M, \partial M)$

⇝ surgery theory.

# §4. Hyperbolic & symmetric Poincaré objects.

Let $V$ findim'l. inner product space over a field $k$

$$V \otimes_k V \xrightarrow{b} k.$$

Suppose $L \subset V$ subspace s.t. $L = L^\perp = \{w : b(w,l)=0 \ \forall l \in L\}$.

Then $L \to V \simeq V^\vee \xrightarrow{i^\vee} L^\vee$ is an exact sequence.

Say $(V,b)$ is <u>metabolic</u> if $\exists\, L \subset V$ satisfying

call $L$ a <u>Lagrangian</u> for $(V,b)$.

Prototypical example:

$$\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & A \end{pmatrix} \qquad A \in \mathrm{Mat}_{n\times n}(k)$$

only equivalent iff $\frac{1}{2} \in k$

"hyperbolic" $(V = L \oplus L^\vee)$.

<u>Motivation</u> $V \rightsquigarrow C_*(M;\mathbb{Z}) \ni x$ s.t. $b(x,x)=0$. $x$ has zero self-intersection $\rightsquigarrow$ can "surgery subspace gen. by $x$ away"

modulo dimension issues.

<u>Def</u> Let $(\mathcal{C},\text{Ϙ})$ Poincaré $\infty$-category w/ duality $D$. Given $x \in \mathcal{C}$, let $\mathrm{hyp}(x)$ the Poincaré object w/:

- underlying object $x \oplus Dx$.
- Poincaré form: image of $\mathbb{1}_x$ under

$$\begin{tikzcd} \mathrm{map}_{\mathcal{C}}(x,x) \arrow[r, "ev_*"] & \mathrm{map}_{\mathcal{C}}(x,DDx) \simeq \Omega^\infty B_{\text{Ϙ}}(x,Dx) \arrow[d] \\ & \Omega^\infty \text{Ϙ}(x \oplus Dx). \end{tikzcd}$$

<u>Def</u> Let $\mathcal{C}$ stable $\infty$-cat. Its <u>hyperbolic category</u> $\mathrm{Hyp}(\mathcal{C})$ is the hermitian $\infty$-cat w/ underlying category $\mathcal{C} \oplus \mathcal{C}^{op}$ + hermitian structure $\text{Ϙ}_{hyp}(x,y) = \hom_{\mathcal{C}}(x,y)$.

↪ Associated symmetric bilinear functor:

$$B_{hyp}((x,y),(x',y')) = \hom_{\mathcal{C}}(x,y') \oplus \underbrace{\hom_{\mathcal{C}}(x',y)}_{\hom_{\mathcal{C}^{op}}(y,x')}.$$

& $L_{Q_{hyp}} \simeq 0$.

↪ $B_{hyp}$ is perfect w/ duality. $D(x,y) = (y,x)$

**Claim** $\mathrm{Hyp}(-)$ defines a functor.
$$\begin{tikzcd} \mathrm{Cat}^{ex}_\infty \arrow[r] & \mathrm{Cat}^{p}_\infty \arrow[l, bend left] \end{tikzcd}$$
left & right adjoint to "forget"

→ using this, can show:

$$\begin{tikzcd} \mathrm{Pn}(\mathrm{Hyp}(\mathcal{C})) \arrow[r] \arrow[rr, bend left, "\sim"] & {}_2\mathcal{C} \oplus {}_2\mathcal{C}^{op} \arrow[r, "p_1"] & {}_2\mathcal{C} \end{tikzcd}$$

---

**Def.** Let $(\mathcal{C}, Q)$ Poincaré $\infty$-category & $(x, q \in \Omega^\infty Q(x))$ Poincaré obj.

An isotropic object over $x$ is a pair

$$(\underset{\text{map in } \mathcal{C}}{f: w \to x}, \; \underset{\text{nullhomotopy in } \Omega^\infty Q(w)}{\eta: f^*q \sim 0})$$

Such an object is Lagrangian if the nullhomotopy of

$$\begin{tikzcd} w \arrow[r] & x \arrow[r, "\simeq"] \arrow[rr, bend right] & Dx \arrow[r] & Dw \end{tikzcd} \in \Omega^\infty B(w,w) \simeq \hom_{\mathcal{C}}(w, Dw)$$

exhibits 🟩 as an exact sequence.

Say that $(x,q)$ is metabolic if it admits a Lagrangian.

**Ex** $\mathrm{hyp}(x)$ is metabolic w/ Lagrangian $x \hookrightarrow x \oplus Dx$.

**Ex** $M$ closed oriented $n$-mfld, $W^{n+1}$ oriented null-bordism of $M$.

$\Rightarrow$ $C^*(W) \to C^*(M)$ + fundamental class.

exhibits $C^*(W)$ as a Lagrangian of $(C^*(M), (-)\cup(-)[M])$.

**Def** Let $(\mathcal{C}, Ϙ)$ Poincaré $\infty$-cat. The associated <u>metabolic category</u> is the hermitian $\infty$-cat $\mathrm{Met}(\mathcal{C}, Ϙ)$ w/

- underlying $\infty$-category $\mathrm{Ar}(\mathcal{C})$.
- quadratic functor $Ϙ_{met}: (w \to x) \mapsto \mathrm{fib}(Ϙ(x) \to Ϙ(w))$.

**Note**: $B_{met}(w \to x, w' \to x') = \mathrm{fib}\left(B_Ϙ(x, x') \to B(w, w')\right)$.

$Ϙ$ perfect w/ duality $D \Rightarrow Ϙ_{met}$ perfect w/ duality

$$D_{met}(w \to x) \longrightarrow \left(\mathrm{fib}(Dx \to Dw) \to Dx\right)$$

**Def** $\mathrm{Pn}^{\partial}(\mathcal{C}, Ϙ) = \mathrm{Pn}(\mathrm{Met}(\mathcal{C}, Ϙ))$

Space of Poincaré objects w/ specified Lagrangian.