Initial definitions

Let $\mathcal{C}$ be an ∞-cat.

- $\mathcal{C}$ is **pointed** if it admits a **zero object** (initial and final)
- $\mathcal{C} \xrightarrow{f} \mathcal{D}$ functor between two ∞-cat is **reduced** if it preserves zeros
- $\mathrm{Fun}_*(\mathcal{C},\mathcal{D}) \subset \mathrm{Fun}(\mathcal{C},\mathcal{D})$ full sub cat. spanned by reduced functors
- $\mathcal{C}$ is **stable** if pointed, admits pushouts and pullbacks and these are the same. Such squares are called **exact**.
- $\mathcal{C} \xrightarrow{f} \mathcal{D}$ reduced functor between stables is **exact** if it preserves pushouts = pullbacks
  $\Leftrightarrow$ preserves finite limits
  $\Leftrightarrow$ preserves finite colimits
- $\mathcal{E} \overset{i}{\subset} \mathcal{D}$ **stable subcategory** if $\mathcal{E}$ is closed under finite (co)limits ($\Rightarrow i$ exact)

$\mathcal{C}$ small

- $\mathrm{Fun}^{ex}(\mathcal{C},\mathcal{D}) \subset \mathrm{Fun}(\mathcal{C},\mathcal{D})$ full sub cat. spanned by exact functors
- $\mathcal{C},\mathcal{D}$ stable $\Rightarrow$ Fun is stable and Fun^{ex}, Fun_* are stable subcats.
- Cat^{ex}_∞ = non-full subcat. of Cat_∞ spanned by stable ∞-cats and exact functors
- $\mathcal{C} \xrightarrow{f} \mathcal{D}$ **1-excisive** if sends pullback to pushout (so in stable = preserves exact □)
- $\mathrm{BiFun}(\mathcal{C}) \subset \mathrm{Fun}_*(\mathcal{C}^{op} \times \mathcal{C}^{op}, Sp)$ full subcat. spanned by **bireduced** functors
  $B(x,y) \simeq 0$ if x or $y \simeq 0$
- $\mathrm{Fun}^b(\mathcal{C}) \subset \mathrm{Fun}(\mathcal{C}^{op} \times \mathcal{C}^{op}, Sp)$ full subcat. spanned by **bilinear functors** (exact in both variables)
- $\mathrm{Fun}^q(\mathcal{C}) \subset \mathrm{Fun}_*(\mathcal{C}^{op}, Sp)$ full subcat. spanned by **quadratic functors** ← red + 2-excisive
- $\mathrm{Fun}^s(\mathcal{C}) := [\mathrm{Fun}^b(\mathcal{C})]^{hC_2}$ ∞-cat. of C_2-equiv. objects in $\mathrm{Fun}^b(\mathcal{C})$ wrt flip action
  Symm bilinear functors on $\mathcal{C}$

vector spaces $V \xrightarrow{f} W$	stable ∞-cats $\mathcal{C} \xrightarrow{f} \mathcal{D}$
$f = (f_i)_i$ s.t. $f_i \in k[x_1, \dots, x_n]$	
• $f \in k[x]_{>0}$	reduced
• $f \in k[x]_{\leq 1}$	1-excisive
	implies
• $f \in k[x]_{\leq 2}$	2-excisive
• $f \in k[x]_{\geq 2 > 0}$	2-excisive + reduced

if $n=1$ this is $bx^2 + ax$

this is what we will care most about

Given $f \in \mathrm{Fun}^{red}(\mathcal{C}^{op} \times \mathcal{C}^{op}, Sp)$ we can always decompose

bireduced replacement

zero when one of x, y is 0

reduced

$$f(x,y) = \underset{(x \cdot y)}{f^{red}(x,y)} + \underset{k[x]_{>0}}{f(x,0)} + \underset{k[y]_{>0}}{f(0,y)}$$

Example:

$\mathcal{C}$ symm monoidal stable ∞-cat. Then $\forall a \in \mathcal{C}$ we have a bilinear functor

$$B_a(x,y) = \hom_{\mathcal{C}}(x \otimes y, a)$$

E.g. take Mod_E for E a ring spectrum. $- \otimes y$, $x \otimes -$ and $\hom(-, a)$ all preserve cofiber sequences.

Def: $Ϙ \in \mathrm{Fun}_*(\mathcal{C}^{op}, \mathrm{Sp})$. Let $B_Ϙ := Ϙ(- \oplus -)^{red}$ be the **cross-effect** of $Ϙ$

Think cross terms

$$f(x) = x^2 + 2x$$

$$\rightsquigarrow \quad f(x+y) = x^2 + y^2 + 2xy + 2(x+y)$$

$$\rightsquigarrow \quad B_f(x,y) = 2xy$$

Call $Ϙ$ **quadratic** if it is additionally 2-excisive $\rightsquigarrow$ ("$ax^2 + bx$")

Remark: The cross effect refines to $\mathrm{Fun}^q(\mathcal{C}) \xrightarrow{B_{(-)}} \mathrm{Fun}^s(\mathcal{C})$

$B_Ϙ \in \mathrm{Fun}^s(\mathcal{C})$ symmetric bilinear part of $Ϙ$

Harvard Thursday

L-theory and GW_0 for Poincaré ∞-categories

One goal of the papers we are reading:

Compute $GW_*(\mathbb{Z})$-groups in a range

why care ?

- GW has a natural occurence in $SH(S)$
- there is an exact sequence

$$K \longrightarrow GW \longrightarrow L$$

alg. K-theory would love to compute

L-theory has been computed

- There is many more, but I'll take those two

The Plan

- Briefly remind of Poincaré ∞-cat notions
- See how the classical study of symm. bil. forms is a precedent
- play around with symm. bil. forms over rings (def. $W(R)$, $GW(R)$, compute examples)
- use $\uparrow$ to motivate def. of L-theory and GW_0 of Poincaré ∞-cats
- Show 1st witness of the seq. $K \to GW \to L$

Notation:

- = notation, comment, ...
- = definition
- = example/mental guideline
- = property
- = summary/result

Remember?

Def: A pair $(\mathcal{C}, Ϙ)$ is called a **hermitian ∞-cat.**

small stable ($\mathcal{C}$); quadratic ($Ϙ$) = hermitian structure on $\mathcal{C}$

A **hermitian functor** $(\mathcal{C}, Ϙ) \to (\mathcal{C}', Ϙ')$ is an exact functor and a nat. transf. $Ϙ \overset{\eta}{\Rightarrow} f^*Ϙ' := Ϙ' \circ f^{op}$ $\leadsto$ denote (f, η)

$Ϙ'(f(x)) \leftarrow Ϙ(x)$

Remark: We have a nat. equiv. $(f \times f)^* B_{Ϙ'} \simeq B_{f^*Ϙ'}$ $\leadsto$ η def. nat. transf.

$$B_Ϙ \Rightarrow (f \times f)^* B_{Ϙ'}$$

$\beta(x, y) \to \beta'(f(x), f(y))$

Def: $(\mathcal{C}, Ϙ)$ hermitian is called Poincaré if $B_Ϙ$ is **perfect**, i.e.

- **nondegenerate** $\leadsto$ $B_Ϙ(x, -)$, $B_Ϙ(-, y)$ are representable
  $\hookrightarrow$ get functor $\mathcal{C}^{op} \xrightarrow{D} \mathcal{C}$ s.t. $B_Ϙ(-, y) = \hom_{\mathcal{C}}(-, Dy)$

$ϙ$ symm. $\rightsquigarrow$ $B(x,y) \simeq \hom_{\mathcal{C}}(x, Dy)$
$\simeq \hom_{\mathcal{C}^{op}}(D^{op}(x), y)$

$\rightsquigarrow$ D^{op} left adjoint to D

$ev: \mathrm{id} \Rightarrow D^{op}D$ unit of adjunction

$\mathrm{id} \Rightarrow D D^{op}$

- ev is an equivalence

$X \simeq X^{\vee\vee}$, i.e. everything is reflexive

$D: \mathrm{Mod}_R^{op} \to \mathrm{Mod}_R$

$X \mapsto X^{\vee}$

$D^{op}D = (\ )^{\vee\vee}$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "ev"'] & Y \arrow[d, "ev"] \\ X^{\vee\vee} \arrow[r, "f^{\vee\vee}"] & Y \end{tikzcd}$$

ev_X

$x \in \mathcal{C},\ q \in \Omega^\infty ϙ(x)$

Def A hermitian form (x,q) is Poincaré if $q_\#: x \xrightarrow{\simeq} D_ϙ(x)$.

$X \cong X^{\vee}$ via chosen pairing

$q_\#$ = image of q under $\Omega^\infty ϙ(x) \to \Omega^\infty B(x,x) = \mathrm{Hom}_{\mathcal{C}}(x, D_ϙ(x))$

Example: $\mathcal{C} = D^p(R)$ perfect complexes ("bdd chains, f.g. proj.")

$B_R(X,Y) = \hom_R(X \otimes_R Y, R)$ is

- symm.
- bilinear
- perfect: $D_R(Y) = \hom_R(Y,R)$

internal mapping complex

Poincaré structures:

$ϙ^s_R(X) := B_R(X,X)^{hC_2}$ $\qquad$ $ϙ^q_R(X) := B_R(X,X)_{hC_2}$

Symm. forms on R $\qquad$ quadratic forms on R

This is a derived version of the classical study of bilinear forms.

bilinear forms

(classical story for motivation + examples)

R = comm. ring

$\mathcal{C} = \mathrm{Mod}_R^{f.g., proj.}$

$$B_R(X,Y) = \mathrm{Hom}_R(X \otimes_R Y, R)$$

$$Q_R^s(X) = B_R(X,X)^{C_2}$$

symm. bilinear forms on X

$X = (X, \beta_X)$ = bilinear pairings $X \otimes_R X \xrightarrow{\beta} R$ s.t. $\beta(x,y) = \beta(y,x)$

A symm. bil. form = (X,β) is an inner product space if β is perfect

A map of forms $X \xrightarrow{f} Y$ is an R-module map which respects the pairings: $\beta_Y(f(x), f(y)) = \beta_X(x,y)$ (isometry)

symm Poincaré object!

β perfect (Poincaré) $\rightsquigarrow$ duality $\beta_\#: X \xrightarrow{\cong} X^\vee = DX$

$x \longmapsto \beta(x,-)$

Example 1

$X = R$. $\rightsquigarrow$ $\mathrm{Hom}_R(R \otimes_R R, R) \cong R$

$\rightsquigarrow$ (X,β) bilinear form, then $\beta = x a$ for some $a \in R$

Notation: $(X, \beta) =: \langle a \rangle$

- Clearly, all forms are symm. (R is comm.).
- $\langle a \rangle$ is inner product iff $a \in R^\times$

- maps $\langle u\rangle \xrightarrow{f} \langle v\rangle$ are elements $\alpha_f \in R$ s.t.

  $(\alpha_f x)u(\alpha_f y) = xuy \rightsquigarrow \alpha_f^2 u = v$ and f iso. $\Leftrightarrow \alpha_f \in R^\times$

$\rightsquigarrow$ $R^\times / R^{2\times}$ represents inner product forms on R

Example 2

$X = R^n$. $\rightsquigarrow$ $\mathrm{Hom}_R(X \otimes_R X, R) \cong \mathrm{Mat}_{n\times n}(R)$

$\rightsquigarrow$ (X, β) bil. form $\rightsquigarrow$ β is given by a square matrix $B = B_X$

and $\beta(x,y) = x^t B y$

Notation: $(X, \beta) =: \langle B\rangle$.

- X ε-symmetric $\Leftrightarrow$ B is ε-symmetric
- maps $\langle B\rangle \xrightarrow{f} \langle B'\rangle$ are matrices $\underset{A_f}{\overset{}{A}}$ s.t. $A^t B A = B'$
- f iso. iff A_f is invertible.

Example 3

$R = \mathbb{R}$. Any f.g. symmetric form module X is (by the spectral theorem) is isomorphic to $\langle \Lambda\rangle$ with Λ diagonal (eigenvalues of B_X).

Any $x \in \mathbb{R}_{>0}$ has a square root. $\langle u\rangle \cong \frac{1}{\sqrt{u}} \langle \mathrm{Sgn}(u)\rangle \frac{1}{\sqrt{u}}$

$\rightsquigarrow \langle \Lambda\rangle \cong \langle 1\!\!1_i \oplus (0 \cdot 1\!\!1_{n-i-j}) \oplus (-1\!\!1_j)\rangle$

via the isom. $\mathrm{Diag}(|d_i|^{-\frac{1}{2}})$ and reordering via permutations

E.g.

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \cong \begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix} \cong \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & \\ & 1 \end{pmatrix}\begin{pmatrix} 4 & \\ & -4 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \\ & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix}$$

Example 2 shows we can define the invariant (determinant)

$$\det(\langle B \rangle) = \det(B) \in R^{\times}/R^{2\times}$$

for X f.g. free. (Can be generalized)

We also define the rank/dimension of an inner product space as the rank of the underlying module: $\mathrm{rank}((X,B)) := \mathrm{rank}(X)$

$$=: \dim((X,B))$$

Careful: $\mathrm{rank}(\langle B \rangle) \neq \mathrm{rank}\, B$ unless B is invertible

always $\mathrm{rank}(\langle B \rangle) = \dim B$.

Goal: Organize the data of inner product spaces into a ring, the Witt ring $W(R)$.

Witt realized a long time ago that there is a ring structure floating around here somewhere. It will have $\oplus, \otimes, \langle\,\rangle$ as its ring structure

X, Y forms

$\oplus$: $X \oplus Y := (X \oplus Y, \beta_X + \beta_Y)$ orthogonal sum

$\otimes$: $X \otimes Y := (X \otimes_R Y, \beta_X \cdot \beta_Y)$ tensor product

This turns inner product spaces into a semiring.

Not what Witt had in mind!

IDEA 1: Take the Grothendieck group.

$P_n(R) :=$ set of isom. classes of inner product spaces on R

Grothendieck group w.r.t. $\oplus$

Grothendieck-Witt ring $GW(R) := G(P_n(R)) := P_n(R) \times P_n(R) / \sim$

$(a,b) \sim (c,d)$ if $\exists m$ s.t. $m + a + d = m + c + b$

This is naturally a ring via

$[a,b] \cdot [c,d] := [ac + bd, ad + bc]$

$-[a,b] = [b,a]$

Remark: i) $\otimes$ is multiplicative w.r.t rank

$$\mathrm{rank}(X \otimes Y) = \mathrm{rank}(X) \cdot \mathrm{rank}(Y)$$

$\oplus$ is additive " " rank

$$\mathrm{rank}(X \oplus Y) = \mathrm{rank}(X) + \mathrm{rank}(Y)$$

$\rightsquigarrow$ rank descends to a surjective ring map

$$GW(R) \xrightarrow{rk} \mathbb{Z}$$

ii) $\det(A \oplus B) = \det A \cdot \det B$

$\rightsquigarrow$ pointed monoid map

$$(GW(R), \oplus, 0) \xrightarrow{\det} (R^\times / R^{2\times}, \times, 1)$$

$$[a,b] \longmapsto \det(a) \cdot \det(b)^{-1}$$

Example We saw that for $R = \mathbb{R}$ we have for X inner product

$$X \cong \langle 1 \rangle_i \oplus \langle -1 \rangle_{n-i} = \bigoplus_{i=1}^{i} \langle 1 \rangle \bigoplus_{i=1}^{n-i} \langle -1 \rangle .$$

own with i pos. and $n-i$ negative eigenvalues

$\rightsquigarrow$ $GW(\mathbb{R})$ gen. by $\langle 1 \rangle$ and $\langle -1 \rangle$.

Remember, $\det(-)$ lands in $R^\times / R^{2\times} \cong C_2$.

- $\det(\langle -1 \rangle \oplus \langle 1 \rangle) = -1 \neq 1 = \det(0)$

  $\rightsquigarrow$ $n\langle -1 \rangle \neq -n \langle 1 \rangle \quad \forall n \in \mathbb{Z} \setminus 0$

- $\mathrm{rank}(\langle -1 \rangle) = \mathrm{rank}(\langle 1 \rangle) = 1$

$\Big\}$ $\langle 1 \rangle$ and $\langle -1 \rangle$ lin. indep. over $\mathbb{Z}$

So we have iso. $\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\cong} GW(\mathbb{R})$ and one turns this into a ring iso.

$$(1,1) \longmapsto \langle 1 \rangle$$
$$(0,-1) \longmapsto \langle -1 \rangle$$

$\rightsquigarrow$ $\boxed{GW(\mathbb{R}) \cong \mathbb{Z}[C_2]}$.

The def. of $GW(R)$ we just gave might work in the non-derived setting, but we will provide a different one which will make GW fall into the mentioned exact sequence $K_{hC_2} \to GW \to L$

We need to define a couple of things before we can make that happen. Those things are best motivated via the classical picture.

Witt did not know about Grothendieck group.
Instead he noticed:

- in. prod. spaces satisfy cancelation:

$$X \oplus Y \cong X \oplus Z \rightsquigarrow Y \cong Z$$

- a certain class of forms (metabolic) form a monoid ideal in $P_n^{\partial}(R) \subset P_n(R)$

- the quotient is a ring

Metabolic forms are forms that exhibit a somewhat "unexpected" behaviour.

Def: X a form. $x,y \in X$ **orthogonal** if $x \cdot y = 0$.

Given a submodule $N \subset X$ we set $N^{\perp} := \{y \in X \mid x \cdot y = 0 \ \forall x \in N\}$

orthogonal complement

A symmetric inner product space X is called **metabolic** or **split** if there exists a submodule $L \subset X$ s.t. $L = L^{\perp}$. Such an L is called a **Lagrangian**

Notation:
- $n \cdot n = 0$ (n is **isotropic**) for $n \in N$ (N is **isotropic**)
- $N \cdot N = 0$ ($n \cdot m = 0 \ \forall n, m \in N$) ($N$ is **totally isotropic**)

Remark: $N \subset X$ being Lagrangian is equivalent to

$$L \hookrightarrow X \cong X^{\vee} \twoheadrightarrow L^{\vee} \text{ being exact}$$

Example
- $H := h := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is metabolic with

  hyperbolic plane

  Lagrangian: $\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle$ $\quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{t} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$

- More generally, given a module M, we define the **associated hyperbolic space** by $h(M) := M \oplus M^{\vee}$ with form $h_M((x,f),(y,g)) = f(y) + g(x)$

  Lagrangians: $M \longrightarrow M \oplus M^{\vee}$, $M^{\vee} \longrightarrow M \oplus M^{\vee}$

- Given an inner product space X we define $-X := (X, -\beta_X)$

  The space $X \oplus -X$ is metabolic

  Lagrangian: $X \xrightarrow{\Delta} X \oplus -X$

  true for nice rings

- Not every metabolic is hyperbolic:

  $R = \mathbb{F}_2$. Consider $X = \langle \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \rangle$. This is an inner product.

  $L = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle$ is a Lagrangian:

  X is not hyperbolic, i.e. $X \not\cong h(V)$ for some V.

  Since $\operatorname{rank} X = 2$, all candidates must be rank 1. $\rightsquigarrow$ only one left

  $h(\langle 1 \rangle) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \not\cong \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

  every element isotropic

  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is not isotropic

$\square$

Now $W(R) := P_n(R) / \partial P_n(R)$

Witt ring of R

= inner prod. sp. modulo metabolic

Def ∞ Let $(\mathcal{C}, \Omega)$ be a Poincaré ∞-cat. and (X, q) a Poincaré object.

A pair $(W \xrightarrow{f} X, \eta)$ where $\eta : f^* q \sim 0 \in \Omega^\infty \Omega(W)$ is a null-homotopy is called an isotropic object.

think $N \hookrightarrow X$ s.t. restriction $\beta_X|_{N \times N} = 0$, i.e. $N \cdot N = 0$

An isotropic object is a Lagrangian if the null-htpy of

$w \xrightarrow{f} x \simeq Dx \xrightarrow{Df} Dw$ given by the image of η in $\Omega^\infty B_{Ϙ}(w,w) = \mathrm{hom}_e(w, Dw)$

exhibits f as exact

$N \hookrightarrow X \simeq X^\vee \twoheadrightarrow N^\vee$ exact

If (x, q) admits a Lagrangian we call it metabolic.

Examples: • $hyp(x) = x \oplus Dx$ $\hookleftarrow x$ or $Dx \longrightarrow x \oplus Dx$ Same as classically

• Not every metabolic object is hyperbolic: Same as classically

$e = D^p(\mathbb{F}_2)$ Let $V = \mathbb{F}_2^2$ with $b = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ this is Poincaré

and we have the Lagrangian $L = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle \hookrightarrow V \neq hyp(L)$

full subgroupoid $P_n \subset F_m$ of Poincaré ob. of the full subgroupoid $F_m \subset He$ of hermitian objects

Remember, $P_n(e, Ϙ) \in \mathcal{S}$ is the space of Poincaré objects.

$\pi_0 P_n(e, Ϙ)$ is naturally a comm. monoid w.r.t.

$+$: $[x, q] + [x', q'] = [x \oplus x', q \perp q']$

$= (q, q', 0) \in \Omega^\infty Ϙ(x \oplus x')$

$\simeq$

$\Omega^\infty Ϙ(x) \times \Omega^\infty Ϙ(x') \times \Omega^\infty B_{Ϙ}(x, x')$

no crossterms

Before we kill metabolics, we organize them into yet another space of Poincaré objects of some Poincaré category.

Def: $(\mathcal{C}, Ϙ)$ Poincaré. $\mathrm{Met}(\mathcal{C}, Ϙ)$ is the hermitian ∞-cat.

$$(\mathrm{Ar}(\mathcal{C}) \overset{\mathrm{Fun}(\Delta^1, \mathcal{C})}{=}, Ϙ_{met}) \quad \text{with} \quad Ϙ_{met} : \mathrm{Ar}(\mathcal{C})^{op} = \mathrm{Ar}(\mathcal{C}^{op}) \xrightarrow{\mathrm{Ar}(Ϙ)} \mathrm{Ar}(Sp) \xrightarrow{fib} Sp$$

$$[w \xrightarrow{f} x] \longmapsto \mathrm{fib}(Ϙ(x) \xrightarrow{Ϙ(f)} Ϙ(w))$$

metabolic is pair $(w \to x, \eta)$ so let's build it that way

$$[w \xrightarrow{f} x, \eta]$$

$$w \to x \simeq Dx \xrightarrow{Df} Dw$$

{ want fiber to exhibit an exact seq.

$w \to Dx \xrightarrow{D(f)} Dw$

Properties:

- hermitian objects are pairs $([w \xrightarrow{f} x], q)$ with null-htpy $f^*q \overset{\eta}{\simeq} 0 \in \Omega^\infty Ϙ(w)$

  isotropic objects

- Poincaré objects is the above, but now Lagrangian

Def: $\mathcal{P}_n^\partial(\mathcal{C}, Ϙ) := \mathcal{P}_n(\mathrm{Met}(\mathcal{C}, Ϙ))$

Poincaré functor

forget the subspace $N \to X$

Note that we have a forgetful map $\mathrm{Met}(\mathcal{C}, Ϙ) \xrightarrow{met} (\mathcal{C}, Ϙ)$

$$[w \to x] \longmapsto x$$

$\rightsquigarrow \quad \mathcal{P}_n^\partial(\mathcal{C}, Ϙ) \longrightarrow \mathcal{P}_n(\mathcal{C}, Ϙ)$ and image is metabolics.

forget Lagrangian

Lemma (2.3.10) $(\mathcal{C}, Ϙ)$ Poincaré. Then

$$L_0 := \pi_0 \mathcal{P}_n(\mathcal{C}, Ϙ) \,/\, \pi_0 \mathcal{P}_n^\partial(\mathcal{C}, Ϙ) \quad \text{is a group.}$$

← cokernel in comm. monoids

proof: $(X \oplus X, q \perp -q) \xleftarrow{\Delta} X$ is a Lagrangian

$$\rightsquigarrow \quad -[X, q] = [X, -q] \qquad \square$$

Exact same proof as classically.

Def (L-theory) $(\mathcal{C}, Ϙ)$ Poincaré.

$$L_n(\mathcal{C}, Ϙ) := \operatorname{coker}\left[\pi_0 \operatorname{Pn}^{\partial}(\mathcal{C}, Ϙ^{[-n]}) \to \pi_0 \operatorname{Pn}(\mathcal{C}, Ϙ^{[-n]})\right]$$

W(R) and GW(R)

Those two are clearly related. We will see how after some examples.

Example Let $k = \bar{k}$ algebraically closed and $\operatorname{char}(k) \neq 2$. One can show that any inner product space over k has an orthogonal basis, i.e.

$$\langle \beta \rangle \cong \bigoplus_{i=1}^{n} \langle u_i \rangle \quad \text{with } u_i \in k^\times.$$

Since any $u \in k^\times$ has a root $\langle u_i \rangle \cong \langle 1 \rangle$.

$$\rightsquigarrow \quad GW(k) \xrightarrow[\cong]{rk} \mathbb{Z} \qquad\qquad L \to X \to L^\vee$$

Note that any metabolic space has even rank and h is metabolic

$$\rightsquigarrow \quad W(R) \xrightarrow{rk} \mathbb{Z}/2 \quad \text{(for general } R\text{)}$$

$$\rightsquigarrow \quad W(k) \xrightarrow[\cong]{rk} \mathbb{Z}/2 \qquad L_0(\mathcal{D}^p(R), \hom_R(-, R)^{hC_2})$$

Did not need full force of alg. closed. Only existence of squareroots.

k quadratically closed

Example we showed earlier $GW(\mathbb{R}) \cong \mathbb{Z}[C_2]$. Our proof already showed that $P_n(\mathbb{R})$ is gen. by $\langle 1\rangle$ and $\langle -1\rangle$. The only metabolic forms possible now are isomorphic to $n\cdot(\langle 1\rangle + \langle -1\rangle)$.

Note when we proved $W(\mathbb{R})$ is a group we showed $-[x,u] = [x,-u]$

$$\rightsquigarrow \quad -\langle 1\rangle = \langle -1\rangle \quad \rightsquigarrow \quad W(\mathbb{R}) \cong \mathbb{Z}.$$

$-(1,1) \qquad (0,-1)$

Example Let k be a perfect field and $\mathbb{S} \in SH(k)$ the sphere spectrum of the motivic stable homotopy category over k.

We have

$$\pi_{0,0}(\mathbb{S}) \cong GW(k)$$

$$\pi_{n,n}(\mathbb{S}) \cong W(k) \quad \text{for } n > 0$$

We have spectra $H_W\mathbb{Z}$, $H\widetilde{\mathbb{Z}}$, $H\mathbb{Z}$ and a pullback square

$$\begin{tikzcd} H_{uW} \arrow[r] \arrow[d] & H_W \arrow[d] \\ H\mathbb{Z} \arrow[r] & H\mathbb{Z}/2 \end{tikzcd} \quad \text{which on } \pi_{0,0} \text{ exhibits} \quad \begin{tikzcd} GW(k) \arrow[r] \arrow[d, "rk"] & W(k) \arrow[d, "rk"] \\ K_0^M(k) \arrow[r, "\mathrm{mod}\,2"] & K_0^M(k)/2 = \mathbb{Z}/2 \end{tikzcd}$$

$$K_0^M(k) \cong \mathbb{Z}$$

and we have a cofiber sequence

$$\begin{tikzcd} H\mathbb{Z} \arrow[r, "h"] & H\widetilde{\mathbb{Z}} \arrow[r] & H_W\mathbb{Z} \end{tikzcd} \quad \overset{\pi_{0,0}}{\rightsquigarrow} \quad \begin{tikzcd} K_0^M(k) \arrow[r, "h"] & GW_0(k) \arrow[r] & W(k) \end{tikzcd}$$

All witnesses of $K_{C_2} \to GW \to L$.

All examples showed $W(R) \cong GW(R)/_{(\text{hyperbolics})}$

In all examples hyperbolics = metabolics unfortunately.

Lemma (Knebusch)

X inner product space with Lagrangian L. Then

$$X \oplus (-X) \cong h(L) \oplus (-X)$$

$\leadsto [X] = [h(L)]$ in $GW(R)$.

$\leadsto [Y] + [-Y] = 0$ in $GW(R)/_{(\text{hyperbolics})}$ for any Y.

proof w.l.o.g. $X \cong L \oplus L^\vee$ with $\beta_X = \begin{pmatrix} 0 & 1 \\ 1 & \beta \end{pmatrix}$.

Now

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ \beta & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & & \\ 1 & \beta & & \\ & & 0 & -1 \\ & & -1 & -\beta \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 1 & \beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & -1 & -\beta \end{pmatrix}$$

$$X \oplus (-X) \xrightarrow{\cong} h(L) \oplus -X$$

Taking $L: Y \xrightarrow{\Delta} Y \oplus (-Y)$ we get

$$[Y] + [-Y] + [-Y] + [Y] = [h(\Delta)] + [-Y] + [Y]$$

$$\leadsto \quad [Y] \equiv -[-Y] \quad (\text{mod hyperbolics}) \quad \square$$

Remark: We have $GW(R) \twoheadrightarrow GW(R)/_{met} \cong W(R)$.

The previous lemma says $[met] = [hyp]$ in GW.

$$\rightsquigarrow \boxed{W(R) \cong GW(R)/_{met} = GW(R)/_{hyp}}.$$

The right definition for the derived setting is imposing $X \sim h(L)$ (as lemma suggests).

Def: $(\mathcal{C}, Ϙ)$ Poincaré ∞-category.

$$\boxed{GW_0(\mathcal{C}, Ϙ) := \pi_0 \, Pn(\mathcal{C}, Ϙ)/\sim}$$

with $[X, q] \sim [hyp(W)]$ for every Lagrangian $W \to X$.

Lemma This is a commutative group.

proof:

- $Z' \to Z \to Z''$ exact in $\mathcal{C}$ $\rightsquigarrow$ $[hyp(Z)] \overset{\text{in } GW_0}{\sim} [hyp(Z')] + [hyp(Z'')]$

  $\Uparrow$ $L := Z' \oplus DZ'' \to hyp(Z)$ is Lagrangian and $hyp(L) \simeq hyp(Z) \oplus hyp(Z'')$

- $\underbrace{[X, q] + [X, -q]}_{= -[X,q]} + [hyp(\Omega X)] \sim [hyp(X)] + [hyp(\Omega X)] \sim 0$

  $[X,q] + [X,-q] \sim [hyp(X)]$ b.c. $X \xrightarrow{\Delta} X \oplus -X$ is Lagrangian

  $\Omega X \to 0 \to X$ is exact

$\square$

Remark: We needed the derived setting to have the relation $[h(\Omega x)] = [h(x)]$!

L-theory and GW_0

We want an exact sequence $K_0(\mathcal{C})_{C_2} \to GW_0(\mathcal{C}, Ϙ) \to L_0(\mathcal{C}, Ϙ) \to 0$

We have seen that the kernel of $GW_0(\mathcal{C}, Ϙ) \longrightarrow L_0(\mathcal{C}, Ϙ) = W$

is classically given by hyperbolics. This generalizes!

Met $\rightleftarrows$ Hyp

We study the interplay between metabolics and hyperbolics.

Hyp, just like Met, organizes into a Poincaré ∞-cat

Def $\mathrm{Hyp}(\mathcal{C}) = (\mathcal{C} \oplus \mathcal{C}^{op}, Ϙ_{hyp})$ with $Ϙ_{hyp}(x,y) = \hom_{\mathcal{C}}(x,y)$

hyperbolic category

symm. bil func
$B_{hyp}((x,y),(x',y'))$
$= \hom(x,y') \oplus \hom(x',y)$

$\begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}$

Properties:

- Always Poincaré: $D_{hyp}(x,y) = (y,x)$
- $P_n(\mathrm{Hyp}(\mathcal{C})) \xrightarrow{\simeq} \iota\mathcal{C}$
  $[(x,y), q] \longmapsto x$

encodes hyperbolic objects and $x \mapsto x \oplus Dx \mapsto x$ is bij.

- C_2-action: "$(x,y) \longmapsto (Dy, D^{op}x)$"

We now have Poincaré functors

where does the name come from?

$$\mathrm{Hyp}(\mathcal{C}) \xrightarrow{\mathrm{can}} \mathrm{Met}(\mathcal{C}, \Qoppa)$$
$$(X, Y) \longmapsto (X \to X \oplus DY)$$

lagrangian

$$\mathrm{Met}(\mathcal{C}, \Qoppa) \xrightarrow{\mathrm{lag}} \mathrm{Hyp}(\mathcal{C})$$
$$(W \to X) \longmapsto (W, D\mathrm{cof}[W \to X])$$

give metabolics on $\mathcal{P}_n$ give lagrangians

$$\mathrm{Hyp}(\mathcal{C}) \xrightarrow{\mathrm{dcan}} \mathrm{Met}(\mathcal{C}, \Qoppa)$$
$$(X, Y) \longmapsto (DY \to X \oplus DY)$$

$$\mathrm{Met}(\mathcal{C}, \Qoppa) \xrightarrow{\mathrm{dlag}} \mathrm{Hyp}(\mathcal{C})$$
$$(W \to X) \longmapsto (\mathrm{cof}[W \to X], DW)$$

herm. structures

$\hom_{\mathcal{C}}(X, Y) \simeq B_{\Qoppa}(X, DY)$

$\simeq \mathrm{fib}[\Qoppa(X \oplus DY) \to \Qoppa(X) \oplus \Qoppa(DY)]$

$\mathrm{fib}[\Qoppa(X \oplus DY) \to \Qoppa(X)]$ $\quad$ $\mathrm{fib}[\Qoppa(X \oplus DY) \to \Qoppa(DY)]$

$\mathrm{fib}[\Qoppa(X) \to \Qoppa(W)]$

$\downarrow$

$\mathrm{fib}[B_{\Qoppa}(X, W) \to B_{\Qoppa}(W, W)]$

$\simeq B_{\Qoppa}(\mathrm{cof}[W \to X], W)$

$\mathrm{lag}^* B_{\mathrm{Hyp}}$ $\quad$ $\mathrm{dlag}^* B_{\mathrm{Hyp}}$

$GW_0(\mathrm{Hyp}(\mathcal{C})) \cong K_0(\mathcal{C})$:

$$\mathcal{P}_n(\mathrm{Hyp}) \simeq \iota\mathcal{C} \rightsquigarrow \pi_0 \mathcal{P}_n(\mathrm{Hyp}) \simeq \pi_0 \iota\mathcal{C}$$

$$(\mathrm{ev}^{\mathrm{op}}_{\iota}, \mathrm{hn}) \ni ((Z, Z), \mathrm{id}_Z) \longmapsto Z$$

GW relations on $\mathcal{P}_n(\mathrm{Hyp})$: metabolic object on $((Z, Z), \mathrm{id})$ is a pair $Z' \to Z$, $Z \to Z''$ s.t. composite $Z' \to Z \to Z''$ is exact since $hyp(z', z'') = (z' \oplus z'', z' \oplus z'') \in \mathcal{C} \times \mathcal{C}^{op}$

$\underset{rel}{\overset{GW}{\leadsto}}$ $[z] = [z'] + [z'']$

Back to our sequence

As classically $GW_0(\mathcal{C}, \mathfrak{L}) \twoheadrightarrow L_0(\mathcal{C}, \mathfrak{L})$

$$L_0(\mathcal{C}, \mathfrak{L}) \cong GW_0(\mathcal{C}, \mathfrak{L}) / met$$

The Poincaré functor $Hyp(\mathcal{C}) \xrightarrow{hyp} \mathcal{C}$ induces $K_0(\mathcal{C}) \xrightarrow{[hyp]} GW_0(\mathcal{C}, \mathfrak{L})$

$x \longmapsto x \oplus Dx$

and a pushout square in comm. monoids

$$\begin{tikzcd} {[w \to x]} \arrow[r, mapsto] & x \\ \pi_0 Pn^\partial(\mathcal{C}, \mathfrak{L}) \arrow[r, "{[met]}"] \arrow[d, "{[\partial]}"'] & \pi_0 Pn(\mathcal{C}, \mathfrak{L}) \arrow[d] \\ K_0(\mathcal{C}) \arrow[r, "{[hyp]}"] & GW_0(\mathcal{C}, \mathfrak{L}) \\ w \arrow[r, mapsto] & w \oplus Dw \end{tikzcd}$$

(by universal properties)

$$\begin{tikzcd} \pi_0 Pn^\partial & {[w \to Dx]} \arrow[d, mapsto] \\ \pi_0 (\mathcal{C}) \arrow[u, leftarrow] & w \\ K_0(\mathcal{C}) \arrow[u, leftarrow] & \end{tikzcd}$$

$\leadsto$ $L_0(\mathcal{C}, \mathfrak{L}) \cong coker(hyp)$ $\qquad$ $W \cong GW/_{hyp}$

$\leadsto$ $K_0(\mathcal{C})_{C_2} \longrightarrow GW_0(\mathcal{C}, \mathfrak{L}) \longrightarrow L_0(\mathcal{C}, \mathfrak{L}) \longrightarrow 0$

[hyp] is C_2-equiv.
(trivial) action
on GW_0