Recollections:

- $(\mathcal{C}, Ϙ)$, $Ϙ : \mathcal{C}^{op} \longrightarrow Sp$

quadratic.

- $Ϙ$ is nondegen. if

$$B_{Ϙ}(X, Y) \simeq \underline{\hom}_{\mathcal{C}}(X, D_{Ϙ} Y)$$

$$D_{Ϙ} : \mathcal{C} \longrightarrow \mathcal{C}^{op}.$$

- $id \Longrightarrow D_{Ϙ} \circ D_{Ϙ}$ if this is an $\simeq$ then say that $(\mathcal{C}, Ϙ)$ is Poincaré

- $(f, \eta) : (\mathcal{C}, Ϙ) \longrightarrow (\mathcal{D}, Φ)$

is the data of

1) an exact functor

$$f : \mathcal{C} \longrightarrow \mathcal{D}$$

2) a transformation.

$$\begin{tikzcd} \mathcal{C}^{op} \arrow[rr, "f^{op}"] \arrow[dr, bend right, "\Upsilon"'] & \arrow[d, phantom, "\Rightarrow"] & \mathcal{D}^{op} \arrow[dl, bend left, "\Phi"] \\ & Sp & \end{tikzcd}$$

$$\Upsilon \Longrightarrow f^* \Phi.$$

Such a datum is $\boxed{\text{Poincaré}}$ as soon as then duality is preserved.

$$f \circ D_{\Upsilon} \xrightarrow{\sim} D_{\Phi} \circ f^{op}.$$

Examples $A = \mathbb{E}_1$-ring

$$\mathcal{C} = \mathrm{Perf}_A \subseteq \mathrm{Mod}_A$$

$$\begin{tikzcd} \Upsilon(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \underline{\hom}_A(X, N) \arrow[d] \\ B_{\Upsilon}(X,X) \arrow[r] & \underline{\hom}_A(X, M^{tC_2}). \end{tikzcd}$$

$\simeq \underline{\hom}_{A\otimes A}(X \otimes X, M)^{hC_2}$

where $A \longrightarrow (A\otimes A)^{tC_2}.$

$M \in \mathrm{Mod}^{hC_2}_{A\otimes A}$

<u>Def</u>. An A-mod. w/ <u>gen inv.</u>

is the data $\left(N \xrightarrow{\alpha} \underbrace{M^{tC_2}}_{A\text{-module.}} \right)$

$M \in \mathrm{Mod}^{hC_2}_{A\otimes A}$

<u>Fact</u>: $Ϙ^{\alpha}_{M}$ nondegen $\iff$ M is perfect as an A-mod

$A \longrightarrow A \otimes A$.

$Ϙ^{\alpha}_{M}$ Poincaré $\iff$ $A \xrightarrow{\sim} \underline{\hom}_A(M, M)$

($M =$ "shifted line bundle")

<u>Example of a Poincaré functor</u>

$A \longrightarrow B$ $\mathbb{E}_1$-map

$\mathrm{Perf}_A \longrightarrow \mathrm{Perf}_B$

$$X \longmapsto B \otimes_A X.$$

LHS: $(M_A \xrightarrow{\alpha} N_A^{t(_2})$

RHS: $(M_B \xrightarrow{\beta} N_B^{t(_2})$

I.3.3.2 the transformation

$$\varphi^{\alpha}_{N_A} \Longrightarrow f^* \varphi^{\beta}_{N_B}.$$

$\Longleftrightarrow \left\{ \begin{array}{l} (\delta, \gamma, \sigma). \\ \delta : N_A \longrightarrow N_B \in \mathrm{Mod}^{hC_2}_{A \otimes A}, \\ \gamma : M_A \longrightarrow M_B \in \mathrm{Mod}_A \\ + \end{array} \right\}$

$$\begin{tikzcd} N_A \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow \sigma"] & N_B \arrow[d] \\ M_A^{tC_2} \arrow[r] & M_B^{tC_2} \end{tikzcd}$$

I.3.3.3 the map of Poincaré cat's is

Poincaré iff.

$$\begin{tikzcd} B \otimes_A M_A \arrow[r] \arrow[rr, bend right] & (B \otimes_S B) \otimes_{(A \otimes_S A)} M_A \arrow[r, "\delta"] & M_A \end{tikzcd}$$

Composite is an equivalence.

* this condition is purely quadratic.

"Pf"

$$B \otimes_A D_{M_A}(X) \longrightarrow D_{M_B}(B \otimes_A X)$$

$$\wr\wr$$

$$\simeq B \otimes_A \underline{\hom}_A(X, M_A) \longrightarrow \underline{\hom}_B(B \otimes_A X, M_B)$$

X perfect, so reduce everything to

X = A

<u>Ex</u>: $A = \mathbb{E}_1$-alg w/ anti-inv.

$\in \mathrm{Alg}_{\mathbb{E}_1}^{hC_2}$ $(A \longmapsto A^{op})$

$\Leftrightarrow$ $\overline{(\cdot)} : A^{op} \longrightarrow A$.

$A \otimes A^{op} \curvearrowright A$

$(-) \uparrow 2$

$A \otimes A$

we can look at the Poincaré functor on

Perf_A given by

$(N \longrightarrow A^{tC_2})$

Give a map $\varphi : A \longrightarrow B$ ring map

of $\mathbb{E}_1$-rings w/ anti-inv's

and $\mathfrak{I}_A : N_A \longrightarrow {}_A(N_B)$ map of + compatibility.

A-modules then we get an induced

functor $(\mathrm{Perf}_A, \mathfrak{I}) \longrightarrow (\mathrm{Perf}_B, \mathfrak{I})$

Criterion:

$$\begin{tikzcd} B \otimes_A A \arrow[r] \arrow[rr, bend right, "\simeq"'] & (B \otimes_A B) \otimes_{A \otimes A} A \arrow[r] & B \end{tikzcd}$$

if is !

new

Example [generating a Poincaré str. out of old ones]

$\varphi : A \longrightarrow B$

$$\mathrm{Perf}_A \longrightarrow \mathrm{Perf}_B$$

$$X \longmapsto X \otimes_A B.$$

$$\begin{tikzcd}
\mathrm{Perf}_A^{op} \arrow[r, "f"] \arrow[d, "Ϙ"'] & \mathrm{Perf}_B^{op} \arrow[dl, dashed, "\underline{LKE}"] \\
Sp &
\end{tikzcd}
\qquad Ϙ \Longrightarrow f^* f_! Ϙ.$$

<u>Question</u>: Ϙ given by $(N \xrightarrow{\alpha} M^{tC_2})$.

$\rightsquigarrow$ new Poincaré functor on Perf_B

$\rightsquigarrow$ is the induced transf.

<u>Claim</u>: $f_!(N \xrightarrow{\alpha} M^{tC_2})$ $\qquad f = \varphi^*$

$$\simeq \left(B \otimes_A N \xrightarrow{id \otimes \alpha} B \otimes_A M^{tC_2} \xrightarrow[1 \cdot x]{} ((B \otimes_S B) \otimes_{A \otimes_S A} M)^{tC_2} \right).$$

Some actual examples.

Ex I: 9.

$$\left(0 \longrightarrow M^{tC_2}\right)$$

then pluggin in formula

$$f_!\left(0 \longrightarrow M^{tC_2}\right) \simeq \left(0 \longrightarrow \left(M \underset{A\otimes A}{\otimes} (B\otimes B)\right)^{tC_2}\right)$$

(!) Quadratic forms just base changes.

Ex II: 5.

$$\left(M^{tC_2} \xrightarrow{\text{id}} M^{tC_2}\right)$$

does not go into id.

Ex: $\mathbb{S} \longrightarrow \mathbb{S}[\tfrac{1}{2}]$.

then on the one hand:

$$\mathbb{S}^{tC_2} \simeq \mathbb{S}_2^{\wedge}.$$

$$\mathbb{S}[\tfrac{1}{2}] \otimes_{\mathbb{S}} \mathbb{S}^{tC_2} \simeq \mathbb{S}[\tfrac{1}{2}] \otimes_{\mathbb{S}} \mathbb{S}_2^{\wedge}$$

$$\simeq H\mathbb{Q}_2$$

but $\quad (\mathbb{S}[\tfrac{1}{2}] \otimes_{\mathbb{S}} \mathbb{S})^{tC_2} \simeq 0.$

<u>Ex.</u> [w/ Nardin - Shah - Yang] transfers

- $A \xrightarrow{f} B$ comm disc. rings, f finite <u>flat</u>
  $(\operatorname{Spec} B \xrightarrow{f} \operatorname{Spec} A)$

$$\mathrm{Perf}_B \longrightarrow \mathrm{Perf}_A$$

$$M \longmapsto {}_A M$$

- <u>Recall</u>.

. we have the dualizing ~~sheaf~~ module.

$$\omega_{B/A} = \underline{\hom}_A(B, A) \quad [f^! \mathcal{O}_A]$$

thought of as a B-module.

. if f Gorenstein $\iff$ $\omega_{B/A}$ is

a inv. B-module.

. an <u>orientation</u> is a B-linear iso

$$B \simeq \underline{\hom}_A(B, A)$$

(⇐) Choice $\varphi : B \to A$ A linear such that the form

$$B \times B \longrightarrow A \quad \text{is nondegen.}$$
$$(b_1, b_2) \longmapsto \varphi(b_1 \cdot b_2)$$

Construct:

$$f_* : (\mathrm{Perf}_B, Ϙ^{s}_{f^*M}) \longrightarrow (\mathrm{Perf}_A, Ϙ^{s}_{M})$$

We consider the map.

$$\underline{\hom}_B(N \otimes_B N, f^*M)^{hC_2} \xrightarrow{\varphi_*} \underline{\hom}_A(N \otimes_A N, M)^{hC_2}.$$

Poincaré?

$$\begin{tikzcd}
\underline{\hom}_B(X, f^*M) \arrow[rr] \arrow[d, "\cong"] & & \underline{\hom}_A(f_*X, M) \arrow[ddd, "\cong"] \\
\underline{\hom}_B(X, M \otimes B) \arrow[d, "\boxed{\cong\ \varphi_*}"] & & \\
\underline{\hom}_B(X, M \otimes \underline{\hom}_A(B, A)) \arrow[r, phantom, "\simeq"] & \underline{\hom}_B(X, \underline{\hom}_A(B, M)) &
\end{tikzcd}$$

Goal: understand what it means for

$$(\mathcal{C}, \mathfrak{Ϙ}) \longrightarrow (\mathcal{D}, \Phi) \longrightarrow (\mathcal{E}, \Psi)$$

to be a bifiber (cofiber + fiber) seq in $\mathrm{Cat}_\infty^{p.}$

to <u>anticipate</u>:

<u>3 notions</u>

K-thy

Poincaré

Split Verdier seq. $\rightsquigarrow K$ <u>conn.</u> $\mathrm{GW}^{\mathrm{add}}$

∩|

Verdier GW [paper II]

∩|

Karoubi seq. } Thomason - Neeman - Yao. $\rightsquigarrow \mathbb{K}$ non-conn. (Thomason.)

[Verdier up to idem's]

$\mathbb{GW}$ [not out yet!]

diff. flavors of CW - thy.

<u>Short story</u> about K - thy.

$$K : \mathrm{Cat}_\infty^{ex} \longrightarrow Sp$$

some functor

What is it supposed to do. ?

- $\mathcal{C} \xrightarrow{f} \mathcal{D}$ induces an $\simeq$ in $K(f)$

  Call this a K - $\simeq$.

- $\mathcal{C} \longmapsto 0$ ← $\mathcal{C}$ too big (has all sums) $K(\mathcal{C}) \simeq 0.$

  Call this K-0

-
$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{D} \arrow[d] \\ \mathcal{J} \arrow[r] & \mathcal{E} \end{tikzcd}$$
  squares

Such a diagram call K- cartesian if $K(\square)$ = pullback.

$$\mathcal{C} \longrightarrow \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C} \quad \underline{\text{Verdier}}$$

non-conn. K-thy, at the very least takes the above to a fiber seq of spectra.

But Verdier seq's are kinda dumb

<u>Example</u> [Thomason]

$$\mathcal{C} \longrightarrow \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C}$$

if Verd seq.

But $\quad U \overset{j}{\hookrightarrow} X \hookleftarrow Z = X \setminus U$

$$\mathrm{Perf}(X \text{ on } Z) \longrightarrow \underbrace{\mathrm{Perf}(X) \xrightarrow{j^*} \mathrm{Perf}(U)}_{\text{not usually ess. surj}} \quad [K_{-1}]$$

not quite the correct seq; instead we should be looking at fiber seq after idempotent completing

$\mathrm{Verd}\,\mathcal{D} \subseteq \boxed{\mathrm{Kar}\,\mathcal{D}} \subseteq K\text{-Cart}.$

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{D} \arrow[d] \\ \cdot \arrow[r] & \mathcal{D}/\mathcal{C}. \end{tikzcd}$$

the insight of various people. (Walchwan, Kontsevich, Tabuada, & CT)

- $\mathrm{Perf}_C \longrightarrow D^b_C \longrightarrow MF_C.$

Def $(\mathcal{C}, \text{Ꮙ}) \xrightarrow{(f,\eta)} (\mathcal{D}, \Phi) \xrightarrow{(p,\theta)} (\mathcal{E}, \Psi)$ $(*)$

seq in Cat^p_∞ whose composite is null. Then we say that $(*)$ is a Poincaré-Verdier (PV) seq if it is a fiber + cofiber (bifiber) seq in Cat^p_∞.

In this situation:

. $(f,\eta) : (\mathcal{C}, \mathfrak{I}) \to (\mathcal{D}, \mathfrak{I})$

is called a <u>PV-inclusion</u>.

. $(p,\theta) : (\mathcal{D}, \mathfrak{I}) \to (\mathcal{E}, \mathfrak{I})$

is called a <u>PV-projection</u>.

<u>Def</u>. A Verdier seq is a bifiber seq in $\mathrm{Cat}^{\mathrm{ex}}_{\infty}$:

What means?

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[rr, bend right, "0"'] & \mathcal{D} \arrow[r] \arrow[d, phantom, "\Downarrow\simeq"] & \mathcal{E} \\ & {} & \end{tikzcd}$$

∘ $\square$ is a pullback.

$$\rightsquigarrow \begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{D} \arrow[d, "p"] \\ \{0\} \arrow[r] & \mathcal{E} \end{tikzcd} \iff \mathcal{C} \simeq \mathrm{Ker}(p)$$

$\mathrm{Ker}(p)$ = full subcat of $\mathcal{D}$ s.t. $p(X) = 0$.

. $\square$ is pushout

$\iff \mathcal{D} \to \mathcal{E}$ is

of the form. $D/\mathcal{C}$.

Verdier quotient

Explicitly: $D/\mathcal{C} = \underbrace{D[W^{-1}]}_{\text{"Dwyer-Kan"}}$

! no adjoint i.e. dependence

$W = \{ x \longrightarrow y$ in D w/ fiber in $\mathcal{C} \}$.

II.A.10 Suppose

$(*)$ $\mathcal{C} \xrightarrow{f} D \xrightarrow{p} \mathcal{E}$ seq w/ vanishing composite then tfae.

1) $(*)$ is a Verdier seq.

2) f is fully faithful, image closed under retracts and

$$D \longrightarrow \mathcal{E} \simeq D/\mathcal{C} \simeq D[W^{-1}].$$

so, p exhibits $\mathcal{E}$ as a Verdier quotient of D by $\mathcal{C}$.

- Most nontrivial part: $\mathcal{C} \longrightarrow D$ any functor

$$D \xrightarrow{p} D/\mathcal{C} \quad , \ker(p) \supseteq \text{Ess. Im of } \mathcal{C}$$

but may be bigger, in fact is closure under retracts.

This already buys us something

$$\begin{tikzcd} \mathrm{Cat}^{p}_{\infty} \arrow[d, "fgt"] \\ \mathrm{Cat}^{ex}_{\infty} \arrow[u, bend left, "\mathrm{Hyp}"] \arrow[u, bend right, "\mathrm{Hyp}."'] \end{tikzcd}$$

$$\mathrm{Hyp}(\mathcal{C}) = \mathcal{C} \times \mathcal{C}^{op}$$

- $\hom_{\mathcal{C}} : (\mathcal{C} \times \mathcal{C}^{op})^{op} \longrightarrow Sp.$

- $\mathrm{Hyp} \circ fgt(\mathcal{C}, \text{Ϙ}) \xrightarrow{hyp} (\mathcal{C}, \text{Ϙ})$

$$hyp(X, Y) = X \oplus D_{\text{Ϙ}} Y.$$

Therefore.

<u>II.1.B</u> : • underlying seq of a P-V seq

is Verdier

- Hyp of a V-seq is a P-V-seq.

More unpacking.

II.1.4.1. [Recognition]

$$(\mathcal{C}, \varphi) \xrightarrow{(f,\eta)} (D, \Phi) \xrightarrow{(p,\theta)} (\mathcal{E}, \Psi) \quad (*)$$

seq w/ vanishing composite

1) $*$ is a fiber iff.

$$\mathcal{C} \simeq \ker(p)$$

and the map $\varphi \xrightarrow{\sim} f^*\Phi$ is an $\simeq$ [φ is the restr. of Φ]

2) $*$ cofiber seq iff.

$D \to \mathcal{E}$ is a Verdier quot.

$\theta : \Phi \longrightarrow p^! \Psi$.

$$\begin{tikzcd} D^{op} \arrow[r, "\overline{\Phi}"] \arrow[d, "p^{op}"'] \arrow[dr, phantom, "\Downarrow \theta"] & Sp \\ \mathcal{E}^{op} \arrow[ur, bend right] & \end{tikzcd}$$

$\Phi_{\cdot} \simeq p_! \overline{\Phi}$

[Φ is the LKE of $\overline{\Phi}$ along p^{op}]

$$(\mathcal{C}, \Psi) \xrightarrow{\text{PV-incl}} (D, \overline{\Phi}) \xrightarrow[\text{PV-proj}]{} (\mathcal{E}, \Phi)$$

__Cor.__ $(f, \eta) : (\mathcal{C}, \Psi) \longrightarrow (D, \Phi)$

- is a PV incl iff f f.f., retract closed + $\Psi \to f^* \Phi$.

- is a PV proj iff $D \to \mathcal{E}$ loc + $\overline{\Phi} \to p^* \Phi$ is LKE.

__Examples__ $p : D \to \mathcal{E}$ V-projection

$\overline{\Phi}$ is a Poincaré str. on D.

$\rightsquigarrow$ $p_! \overline{\Phi} : \mathcal{E}^{op} \to Sp$.

__Claim:__ $\overline{\Phi} \longrightarrow p^* p_! \overline{\Phi}$ is Poincaré iff

$\mathrm{Ker}(p)$ is invariant under the duality.

Split versions

Def A Verdier seq is split

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook] & \mathcal{D} \arrow[r, "p"] & \mathcal{E} \arrow[l, bend right, "L"'] \arrow[l, bend left, "R"] \end{tikzcd}$$

Usually a split Verdier seq s.th. that looks like

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook] & \mathcal{D} \arrow[l, bend right] \arrow[r] & \mathcal{E} \arrow[l, bend right, "L"'] \end{tikzcd}$$

Observe.

$$Lp \longrightarrow \underbrace{\mathrm{id} \longrightarrow}_{\text{the unit map of the adj'n.}} \mathrm{cof.} \simeq f \circ L'$$

$$\begin{tikzcd} \mathrm{cof} : \mathcal{D} \arrow[rr] \arrow[dr, bend right, "L'"'] & & \mathcal{D} \\ & \mathcal{C} \arrow[ur, bend right] & \end{tikzcd}$$

$L' : \mathcal{D} \rightleftarrows \mathcal{C} : f.$

"Thm"

$$\begin{tikzcd} \mathcal{C} \arrow[r] & \mathcal{D} \arrow[r] & \mathcal{E} \arrow[l, bend right, "\perp"'] \arrow[l, bend left, "\perp"] \end{tikzcd}$$

then can complete to.

1)

$$\begin{tikzcd} \mathcal{C} \arrow[r] & \mathcal{D} \arrow[r] \arrow[l, bend right, "\perp"'] \arrow[l, bend left, "\perp"] & \mathcal{E} \arrow[l, bend right, "\perp"'] \arrow[l, bend left, "\perp"] \end{tikzcd}$$

2) this the same data as a <u>recollement</u>

$$\begin{tikzcd} \mathcal{C} \arrow[r] & \mathcal{D} \arrow[r, "L_{\mathcal{E}}"] \arrow[l, bend right, "\perp"', "L_{\mathcal{C}}"] & \mathcal{E} \arrow[l, bend left, "\perp"] \end{tikzcd}$$

$\to$ data of

two subcat's

- $\mathcal{E}, \mathcal{C} \hookrightarrow \mathcal{D}$
- such that incl have left adj's, $L_{\mathcal{C}}, L_{\mathcal{E}}$
- $L_{\mathcal{C}}, L_{\mathcal{E}}$ jtly cons.
- $\mathcal{C} \longrightarrow \mathcal{D} \xrightarrow{L_{\mathcal{E}}} \mathcal{E}$

$\parallel$

0

Examples

- $$\begin{tikzcd} Sp[\tfrac{1}{\ell}] \arrow[r] & Sp \arrow[l, bend right, "{[\frac{1}{\ell}]}"'] \arrow[l, bend left, "\mathrm{div}_\ell"] \arrow[r] & Sp[[\ell\text{-adic}-1]^{-1}] \arrow[l, bend right, "{[-][\ell^\infty]}"'] \arrow[l, bend left, "(\ )^\wedge_\ell"] \end{tikzcd}$$

ℓ^∞-tors

ℓ-div part

Exam'l Thm

$$\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$$ vanishing composite

note: [reversed]

then! if p is a localization in V-pres

$$\begin{tikzcd} \mathrm{Ind}(\mathcal{E}) \arrow[r, hook, "\perp"] & \mathrm{Ind}(\mathcal{D}) \arrow[l, bend right, "\mathrm{Ind}(p)"'] \arrow[l, bend left] \arrow[r] & \mathrm{Ind}(\mathcal{C}) \arrow[l, bend right, "\mathrm{Ind}(f)"'] \arrow[l, bend left] \end{tikzcd}$$

assume Verdier incl.

pres. colimits

We say that $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$

is a Karoubi seq is

Ind of it is split <u>Verdier</u>.

! $K(\mathrm{Ind}(h)) = 0$, but

K-exactness can be checked

after Ind.

Back to the Poincaré world.

<u>Def</u>. A Verdier seq is split-Poincaré-Verdier (split PV)

if the underlying seq is split Verdier

<u>Remark</u>. Checking splitness is easier because:

if $L \dashv p$ then $p^{op} \dashv L^{op} \simeq R$

$$\begin{matrix} \| \\ D_q \, p \end{matrix}$$

So a PV seq is split as soon as p has a right or a left adjoint.

<u>Prop</u> Split PV $\Leftrightarrow$ p has a fully

faithful left adjoint

$$(\mathcal{D}, \Phi) \underset{p}{\overset{g}{\longrightarrow}} (\mathcal{E}, \Phi)$$

$$\underbrace{g^* \Phi \Longrightarrow g^* p^* \Phi \Longrightarrow \Phi}_{\simeq}$$

Main.

Example Metabolic sequence.

$$(\mathcal{C}, Ϙ^{[-1]}) \longrightarrow \mathrm{Met}(\mathcal{C}, Ϙ_{met}) \longrightarrow (\mathcal{C}, Ϙ)$$

$$Ϙ^{[-1]} \cong \Sigma^{-1} Ϙ$$

$$\mathrm{Met}(\mathcal{C}, Ϙ) = \mathrm{Ar}(\mathcal{C}) = \{x \to y\}$$

$$Ϙ_{met}(x \to y) \simeq \mathrm{fib}(Ϙ(y) \to Ϙ(x))$$

is split Verdier.

$$\Omega X \longrightarrow 0 \longrightarrow X$$

- $\mathcal{C} \longrightarrow \mathrm{Ar}(\mathcal{C}) \xrightarrow{t} \mathcal{C}.$

$$X \longmapsto X \longrightarrow 0$$

$$X \longrightarrow Y \longmapsto Y.$$

- $Ar(\mathcal{C}) \longrightarrow \mathcal{C}$ is a Verdier projection.

<u>Intuition</u>.

$$\begin{tikzcd}
\mathrm{fib} \arrow[r] \arrow[d] & 0 \arrow[d] \\
X \arrow[r, "f"] \arrow[d] & Y \arrow[d, equal] \\
Y \arrow[r, equal] & Y
\end{tikzcd} \in Ar(\mathcal{C})$$

- In fact can check that a left adj

$$Ar(\mathcal{C}) \longrightarrow \mathcal{C}$$

$$0 \to X \longleftarrow\!\shortmid X$$

- Check that

$$(\mathcal{C}, Ϙ^{[-1]}) \xrightarrow[\text{Poincaré}]{} (\mathrm{Met}(\mathcal{C}), Ϙ_{\mathrm{met}})$$

is a Verdier incl.

$$\checkmark\ \mathcal{I}^{\varphi[-1]}(X) \simeq \Sigma^{-1}\mathcal{I}^{\varphi}(X) \simeq \mathrm{fib}\big(\mathcal{I}^{\varphi}(0) \to \mathcal{I}^{\varphi}(X)\big)$$

$$\simeq \mathcal{I}^{\varphi}_{\mathrm{met}}(X \to 0)$$

Cofiber?

$$g^{*}\mathcal{I}^{\varphi}_{\mathrm{met}}(\cancel{\ \ }\ X)$$

$$\wr$$

$$\mathcal{I}^{\varphi}_{\mathrm{met}}(0 \to X)$$

$$\wr$$

$$\mathrm{fib}\big(\mathcal{I}^{\varphi}(X) \to \mathcal{I}^{\varphi}(0)\big) \simeq \mathcal{I}^{\varphi}_{\mathrm{ln}}(X)$$

$\therefore$ conclude that indeed the metabolic seq is a split PV-seq.

Example

$$\underbrace{(\mathrm{Perf}_X)_Z \longrightarrow \mathrm{Perf}_X}_{} \xrightarrow{j^*} \mathrm{Perf}_U$$

quadratic enhancement of this.

$A \xrightarrow{\varphi} B$ of $\mathbb{E}_1$-rings is a localization if

1) $B \otimes_A B \xrightarrow[\simeq]{m} B$

2) $I = \mathrm{fib} \in \mathrm{Mod}_A$ is perfectly gen $\in \langle (\mathrm{Mod}_A)_B \cap \mathrm{Mod}_A^{\omega}, \text{colimits} \rangle$

Given (M, N, α) which is mod / generic inv. on A then we can Kan extend to get one on B.

$$(\mathrm{Mod}_A^{\omega}, Ϙ_M^{\alpha}) \longrightarrow (\mathrm{Mod}_B^{\omega}, Ϙ_{\varphi_! M}^{\varphi_! \alpha})$$

- wants to be a P-V projection.

  but it's not.

- P-Karoubi projection.

- t small condition to check on

  m (always the set A → B)

  conn

the functor is Poincaré.