Thursday Seminar: L-theory and the fundamental square

Thursday, November 4, 2021 13:29

§1. Isotropy Seperation Square

$$\mathrm{Sp}^{gC_2} := \mathrm{Fun}^{\times}(\mathrm{Span}(C_2), \mathrm{Sp})$$

(E^{gC_2}, E) where E has an C_2 action, $E^{gC_2} \to E$, $E \to E^{gC_2}$.

Thm: (Isotropy Seperation Square) For any such genuine C_2 spectrum

$$\begin{tikzcd} E_{hC_2} \arrow[r] & E^{gC_2} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & \Phi^{C_2} E \arrow[d] \\ & E^{hC_2} \arrow[r] & E^{tC_2} \end{tikzcd}$$

Thm (CDH^+ II, Cor. 3.6.7)

Let $\mathcal{F}: \mathrm{Cat}_\infty^p \to \mathrm{Sp}$ be an additive functor, $(\mathcal{C}, Ϙ) \in \mathrm{Cat}_\infty^p$, then there is a pullback

$$\begin{tikzcd} \mathcal{F}(\mathcal{C}, Ϙ) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & \mathcal{F}^{\mathrm{bord}}(\mathcal{C}, Ϙ) \arrow[d] \\ \mathcal{F}^{\mathrm{hyp}}(\mathcal{C}, Ϙ)^{hC_2} \arrow[r] & {[\mathcal{F}^{\mathrm{hyp}}(\mathcal{C}, Ϙ)^{hC_2}]^{\mathrm{bord}}} \arrow[d, phantom, "\simeq" rotate=90] \\ & \mathcal{F}^{\mathrm{hyp}}(\mathcal{C}, Ϙ)^{tC_2} \end{tikzcd}$$

e.g.

$$\begin{tikzcd} GW(\mathcal{C}, Ϙ) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & L(\mathcal{C}, Ϙ) \arrow[d] \\ K(\mathcal{C})^{hC_2} \arrow[r] & K(\mathcal{C})^{tC_2} \end{tikzcd}$$

$$K(\mathcal{C}, Ϙ)_{hC_2} \to GW(\mathcal{C}, Ϙ) \to L(\mathcal{C}, Ϙ)$$

§2: Bordism invariant functors & bordification.

Defn. Two Poincare functors $f, g : (\mathcal{C}, Ϙ) \to (\mathcal{D}, \Phi)$ are *cobordant* if there exists a

$$h : (\mathcal{C}, Ϙ) \to \underline{\mathcal{Q}_1(\mathcal{D}, \Phi)}$$

s.t. $d_0 h = f$ and $d_1 h = g$.

A cobordism invariant functor is an additive functor

$$\mathcal{F} : \mathrm{Cat}^p_\infty \to \mathrm{Sp}$$

s.t. cobordant maps are htpic.

$$f : (\mathcal{C}, Ϙ) \rightleftarrows (\mathcal{D}, \Phi) : g$$

Ex: 1.)

$$\begin{tikzcd} (\mathcal{C}, Ϙ) \arrow[r, "s"] & \mathcal{Q}_1(\mathcal{C}, Ϙ) \arrow[d, "d_1"] \\ & (\mathcal{C}, Ϙ) \end{tikzcd}$$

These are inverse cobordant equivalences

Lemma: [3.5.4, CDH$^+$ II] If $\mathcal{F}$ is an additive group-like functor then $\mathcal{F}$ is bordism invariant $\iff$ $\mathcal{F}(\mathrm{Hyp}(\mathcal{C})) = 0$.

pf:

$\Rightarrow$) $(\mathcal{C}, Ϙ) \to \mathcal{Q}_1(\mathcal{C}, Ϙ) \to \mathrm{Hyp}(\mathcal{C}, Ϙ)$

is split PV

$$\begin{tikzcd}
\mathcal{F}(\mathcal{C},Ϙ) \arrow[r, "\simeq"] & \mathcal{F}(Q_1\mathcal{C},Ϙ) \arrow[r] & \mathcal{F}(\mathrm{Hyp}(\mathcal{C})) \arrow[d, equal] \\
 & & 0
\end{tikzcd}$$

(⇐.)

$$\begin{tikzcd}
\underline{\mathcal{F}(\mathcal{C},Ϙ)\oplus\mathcal{F}(\mathrm{Hyp}(\mathcal{C}))} \arrow[r, "\mathcal{F}(s)\oplus\mathcal{F}(i_{\mathrm{Hyp}})"] \arrow[dr, bend right, "\simeq"'] & \mathcal{F}(Q_1(\mathcal{C},Ϙ)) \arrow[d, "\mathcal{F}(d_1)\oplus\mathcal{F}(p_{\mathrm{Hyp}})"] \\
 & \underline{\mathcal{F}(\mathcal{C},Ϙ)\oplus\mathcal{F}(\mathrm{Hyp}(\mathcal{C}))}
\end{tikzcd}$$

$\underline{f \xleftarrow{} h \xrightarrow{} g} \longmapsto \mathcal{F}(\underline{N}h) \sim \underline{\mathcal{F}(g)-\mathcal{F}(f)}$

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{bord}}(\mathrm{Cat}^{p}_{\infty},\mathrm{Sp}) \arrow[r] & \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{p}_{\infty},\mathrm{Sp}) \arrow[l, bend right, "\mathrm{bord}"'] \arrow[l, bend left, "\mathrm{cobord}"]
\end{tikzcd}$$

Lemma/Defn. For $\mathcal{F}$ an additive functor

$$\mathcal{F}^{\mathrm{bord}}(\mathcal{C},\underline{Ϙ}) \simeq \mathrm{cof}\left(\mathcal{F}^{\mathrm{hyp}}_{hC_2}(\mathcal{C},Ϙ) \to \mathcal{F}(\mathcal{C},Ϙ)\right)$$

$\mathcal{C} \xrightarrow[\simeq]{D_Ϙ} \mathcal{C}^{op}$

$\mathcal{C}\oplus\mathcal{C}^{op} \xrightarrow{D_Ϙ\oplus D_Ϙ^{op}} \mathcal{C}^{op}\oplus\mathcal{C} \xrightarrow{\mathrm{swap}} \mathcal{C}\oplus\mathcal{C}^{op}$

$$\begin{tikzcd}
\mathcal{F}(\mathcal{C},Ϙ) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{F}^{\mathrm{bord}}(\mathcal{C},Ϙ) \arrow[d] \\
\mathcal{F}^{\mathrm{hyp}}(\mathcal{C},Ϙ)^{hC_2} \arrow[r] & \left[\mathcal{F}^{\mathrm{hyp}}(\mathcal{C},Ϙ)^{hC_2}\right]^{\mathrm{bord}}
\end{tikzcd}$$

$$\mathrm{Nat}(\mathcal{F}^{\mathrm{hyp}}_{hC_2}, \mathcal{G}) \simeq \mathrm{Nat}(\mathcal{F}^{\mathrm{hyp}},\mathcal{G})^{hC_2}$$
$$\simeq \mathrm{Nat}(\mathcal{F},\mathcal{G}^{\mathrm{hyp}})^{hC_2}$$
$$= 0$$

$$\Rightarrow \mathrm{Nat}(\mathcal{F}^{\mathrm{bord}},\mathcal{G}) \simeq \mathrm{Nat}(\mathcal{F},\mathcal{G}).$$

Cor: $\left[\mathcal{F}^{\mathrm{hyp}}(-)^{hC_2}\right]^{\mathrm{bord}} \simeq \mathcal{F}^{\mathrm{hyp}}(-)^{tC_2}$.

§3: ρ construction and ad construction

Let $\mathcal{J}_n := \mathcal{P}_0([n])^{op}$ — non empty subsets of $[n]$

$$\mathcal{P}_0([n])^{op} = \underline{\mathrm{sd}(\Delta^n)^{op}}$$

n-cube \ terminal point.

Defn. For $(\mathcal{C}, Ϙ)$ a Poincaré ∞-cat,

$$\mathcal{P}_n(\mathcal{C}, Ϙ) := (\mathcal{C}^{\mathcal{J}_n}, Ϙ^{\mathcal{J}_n})$$

$$G : \mathcal{J}_n \to \mathcal{C}$$

$$\underline{D_{Ϙ^{\mathcal{J}_n}} G(i) = \lim_{\mathcal{J}_n^{op}/i} D_Ϙ G(-)}$$

$\mathcal{P}_n(\mathcal{C}, Ϙ)$ is a Poincaré if $(\mathcal{C}, Ϙ)$ was, face & degeneracy maps are Poincaré.

$$\mathcal{F} : \mathrm{Cat}_\infty^p \to \mathrm{Sp}$$

$$\mathrm{ad}\,\mathcal{F}(-) := |\mathcal{F}\mathcal{P}_n(-)|$$

Lemma: $\mathcal{F} \Rightarrow \mathrm{ad}\,\mathcal{F}$ gives that $\mathrm{ad}\,\mathcal{F}$ is a bordification of $\mathcal{F}$

"pf:" ETS that

1.) $\mathrm{ad}\,\mathcal{F}(\mathrm{Hyp}(\mathcal{C}))| = 0$

2.) If $\mathcal{F}$ already bordism invariant, then $\mathcal{F} \overset{\simeq}{\Rightarrow} \mathrm{ad}\,\mathcal{F}$

$$\begin{tikzcd}
\mathcal{P}_n(\mathcal{C}, Ϙ) \arrow[r] & (\mathrm{Fun}(\mathcal{P}([n])^{op}, \mathcal{C}), \underline{Ϙ^{tf}}) \arrow[d, phantom, "\simeq" sloped] \\
& \mathrm{Met}^{(n+1)}(\mathcal{C}, Ϙ) \arrow[d] \\
& (\mathcal{C}, Ϙ^{[-1]})
\end{tikzcd}$$

is split PV.

$$\mathcal{F}(\underline{\mathcal{P}_n}(\mathcal{C}, Ϙ)) \to \mathcal{F}(\underset{\simeq}{\underline{\mathrm{Met}^{(n+1)}}}(\mathcal{C}, Ϙ)) \to \mathcal{F}(\underline{\mathcal{C}, Ϙ^{[-1]}})$$

$$\begin{tikzcd}
\mathcal{F}(\mathrm{Hyp}(\mathcal{M}et^{(n)}(\mathcal{C},\text{Ϙ}))) \arrow[d, phantom, "\parallel"] \\
0
\end{tikzcd}$$

$$|\mathcal{F}(\mathcal{Q}_\bullet(\mathcal{C},\text{Ϙ}))| \simeq \mathcal{F}(\mathcal{C},\text{Ϙ}).$$

$$\mathrm{ad}\,\mathcal{F}(\mathrm{Hyp}(\mathcal{C})) \simeq 0.$$

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§4: L-Theory

Defn. For $(\mathcal{C},\text{Ϙ})$ a Poincaré ∞-category, the L-theory space

$$\mathrm{L}(\mathcal{C},\text{Ϙ}) := |\mathrm{Pn}\,\mathcal{Q}_\bullet(\mathcal{C},\text{Ϙ})| = \mathrm{ad}\,\mathcal{GW}(\mathcal{C},\text{Ϙ}).$$

Lemma: Let $(\mathcal{D},\Phi)\to(\mathcal{E},\Psi)$ be a PV projection. Then

$$\begin{tikzcd}
\mathrm{Pn}\,\mathcal{Q}(\mathcal{D},\Phi) \arrow[d] \\
\mathrm{Pn}\,\mathcal{Q}(\mathcal{E},\Psi)
\end{tikzcd}$$

is a Kan fibration.

"pf:" 1.) You first show this for

$$\begin{tikzcd}
\mathrm{Fm}(\mathcal{Q}(\mathcal{D},\Phi)) \arrow[d] \\
\mathrm{Fm}(\mathcal{Q}(\mathcal{E},\Psi)).
\end{tikzcd}$$

2.) If you start w/ Poincaré object you can do surgery to find a new lift which is Poincaré.

1.)

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] & \mathrm{Fm}\mathcal{Q}(\mathcal{D},\Phi) \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & \mathrm{Fm}\mathcal{Q}(\mathcal{E},\Psi)
\end{tikzcd}$$

$$J_n = \mathrm{sd}(\Delta^n)^{op}$$

$$\mathrm{Fm}(\mathcal{D}^{\mathcal{J}_n}, \Phi^{\mathcal{J}_n}) \longrightarrow \mathrm{Fm}(\mathcal{D}^{B_n}, \Phi^{B_n}) \times_{\mathrm{Fm}(\mathcal{E}^{B_n}, \Psi^{B_n})} \mathrm{Fm}(\mathcal{E}^{\mathcal{J}_n}, \Psi^{\mathcal{J}_n})$$

Surjection on π_0

$B_n := \mathrm{sd}(\partial\Delta^n)^{op} \subseteq \mathcal{J}_n$

$G \in \mathrm{Fm}(\mathcal{E}^{\mathcal{J}_n}, \Psi^{\mathcal{J}_n})$

$\tilde{G} \in \mathrm{Fm}(\mathcal{D}^{B_n}, \Phi^{B_n})$

$F \in \mathcal{D}^{\mathcal{J}_n}$ $\qquad F|_{B_n} = \tilde{G}$

$F(\{0, \dots, n\}) := \lim \tilde{G}$

Lemma: Let $(\mathcal{D}, \Phi) \xrightarrow{p} (\mathcal{E}, \Psi)$ be a pV projection, $x \in \mathcal{D}$ and a map $f : y \to p(x) \in \mathcal{E}$. Suppose that for a compact space K

$$\begin{tikzcd}
K \arrow[rr] \arrow[d] & & * \arrow[d] \\
\Omega^\infty \Phi(x) \arrow[r] & \Omega^\infty \Psi(p(x)) \arrow[r] & \Omega^\infty \Psi(y)
\end{tikzcd}$$

then $\exists$ $g : w \to x$ and an identification $p(g) = f$, and

$$\begin{tikzcd}
K \arrow[r] \arrow[d] & * \arrow[d] \\
\Omega^\infty \Phi(x) \arrow[r] & \Omega^\infty \Phi(w)
\end{tikzcd}$$

"**pf:**" $p : (\mathcal{D}, \Phi) \to (\mathcal{E}, \Psi)$ is a Verdier projection on underlying cats.

$\exists$ $\tilde{g} : u \to x$ lifting f.

Fact: $\Psi(y) \simeq \operatorname*{colim}_{c \in \mathcal{C}_{u/-}} \Phi(\mathrm{fib}(u \to c))$

where $(\mathcal{C}, \varphi)$ is the cor. Verdier inclusion.

$$G \in \mathrm{Fm}(\mathcal{E}^{J_n}, \Psi^{J_n})$$

$$\tilde{G} \in \mathrm{Fm}(\mathcal{D}^{B_n}, \Phi^{B_n})$$

$$\underset{i \neq [n]}{F(i) = \tilde{G}(i)}, \qquad \underset{\overset{\shortparallel}{x}}{F([n])} = \lim_{B_n} \tilde{G}(-)$$

$$y = G([n])$$

$$F : y \to p(x)$$

$$\begin{tikzcd}
\lvert B_n \rvert \arrow[rr] \arrow[d] & & * \arrow[d, "q"] \\
\Omega^\infty \Phi(x) \arrow[r] & \Omega^\infty \Psi(p(x)) \arrow[r] & \underline{\Omega^\infty \Psi(y)}
\end{tikzcd}$$

so our lemma gives $F \rightsquigarrow \tilde{F}$ s.t. the form data works out.

2.) $\tilde{F}$ a lift of above

$$\mathrm{Fm}(\mathcal{D}^{J_n}, \Phi^{J_n}) \underline{\simeq} \mathrm{Pn}(\mathrm{Met}(\mathcal{D}^{J_n}, \Phi^{J_n}))$$

$$\tilde{F} \longmapsto \underline{D_{\Phi^{J_n}}} \tilde{F} \to \underline{\mathrm{cof}(\tilde{F} \to D_{\Phi^{J_n}} \tilde{F})}$$

$\tilde{F}$ is Poincaré $\iff$ vanishes.

"this is enough to show that $\mathrm{cof}(\tilde{F} \to D_{\Phi^{J_n}} \tilde{F})$ admits a lagrangian."

$$0 \to L$$

Cor: $\mathcal{L}_i(\mathcal{C}, \varphi) \cong \pi_0 \mathcal{L}(\mathcal{C}, \varphi^{[-i]})$

pf: Since this is a Kan complex, you can express the homotopy groups as simplicial homotopy groups.

$$[L \to x] \longrightarrow x$$

$$\underline{\left[(\mathcal{C}, \varphi^{[1]}) \to \mathrm{Met}(\mathcal{C}, \varphi) \to (\mathcal{C}, \varphi)\right]}$$

PV sequence

$$\mathcal{L}(\mathcal{C}, \varphi^{[1]}) \to 0 \to \mathcal{L}(\mathcal{C}, \varphi)$$

$$\implies \mathcal{L}(\mathcal{C}, \varphi^{[1]}) \simeq \Omega \mathcal{L}(\mathcal{C}, \varphi). \quad \blacksquare$$

$$\boxed{\mathcal{L}_0(\mathcal{C},\Qoppa) := \operatorname{coker}(\pi_0 \underline{Pn}^{\partial}(\mathcal{C},\Qoppa) \to \pi_0 Pn(\mathcal{C},\Qoppa))}$$

Thm/Defn. $L(\mathcal{C},\Qoppa) := \mathcal{C}ob^{\mathcal{L}}(\mathcal{C},\Qoppa)$ is the bordification of $GW(\mathcal{C},\Qoppa)$

$$L(\mathcal{C},\Qoppa) = [\mathcal{L}(\mathcal{C},\Qoppa), \mathcal{L}(\mathcal{C},\Qoppa^{[1]}), \text{—}]$$

$$\Omega^{\infty-i} L(\mathcal{C},\Qoppa) \simeq \mathcal{L}(\mathcal{C},\Qoppa^{[i]}).$$

§3 : Genuine Picture

Let $g^{Hyp} : Cat_{\infty}^{p} \longrightarrow Fun^{\times}(Span(C_2), Cat_{\infty}^{p})$

$$(\mathcal{C},\Qoppa) \longmapsto (\underline{(\mathcal{C},\Qoppa)}, \underline{Hyp(\mathcal{C})})$$

$$\begin{tikzcd}
Cat_{\infty}^{p} \arrow[r] \arrow[rr, bend right, "KR"'] & Fun^{\times}(Span(C_2), Cat_{\infty}^{p}) \arrow[r, "GW"] & Fun^{\times}(Span(C_2), Sp) \arrow[d, phantom, "\wr\wr"] \\
 & & Sp^{gC_2}
\end{tikzcd}$$

$$KR(\mathcal{C},\Qoppa)^{gC_2} = GW(\mathcal{C},\Qoppa)$$

$$u^{*}KR(\mathcal{C},\Qoppa) = GW(Hyp(\mathcal{C})) = K(\mathcal{C})$$

$$\begin{tikzcd}
GW(\mathcal{C},\Qoppa) \arrow[r] \arrow[d] & L(\mathcal{C},\Qoppa) \arrow[d] \\
K(\mathcal{C},\Qoppa)^{hC_2} \arrow[r] & K(\mathcal{C},\Qoppa)^{tC_2}
\end{tikzcd}$$

agrees w/ Fundamental Fiber square.

Cor: $KR(\mathcal{C},\Qoppa^{[1]}) \simeq \Sigma^{1-\sigma} KR(\mathcal{C},\Qoppa)$

Cor: Let R be a complex oriented $\mathbb{E}_1$-ring, M an invertable R-mod, $c \subseteq K_0(R)$ a subgroup closed under involution by M. Then

$$KR(Mod_R^c, \Qoppa^{s}_{-M}) \simeq \Sigma^{2-2\sigma} KR(Mod_R^c, \Qoppa^{s}_{M}).$$

and

$$KR(Mod_R^c, \Qoppa^{q}_{-M}) \simeq \Sigma^{2-2\sigma} KR(Mod_R^c, \Qoppa^{q}_{M})$$

and if $R \in Sp_{\geq 0}$,

$$KR(Mod_R^c, \Qoppa^{\geq m+1}_{-M}) \simeq \Sigma^{2-2\sigma} KR(Mod_R^c, \Qoppa^{\geq m}_{M}).$$