# Introduction to topology

Elden Elmanto

Department of Mathematics, University of Toronto, 40 St. George St., Toronto, ON, M5S 2E4, Canada

*E-mail address*: elden.elmanto@utoronto.ca

*URL*: https://www.eldenelmanto.com/

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CHAPTER 1

# Introduction

These are notes for MATH27H3, the UTSC version of a first class in point set topology.

CHAPTER 2

# Topological spaces

Most of this present chapter acquaints us with the basic syntax of point-set topology. We will also give a plethora of examples to illustrate how the basic ideas of point-set topology works. The basic object of study in point-set topology are topological space — these are sets equipped with the notion of "closeness" which is not quite as fine as the notion of "distance" (as formalized by a metric space) — and continuous functions between them. There is no particularly deep result that we will discuss in this chapter, but the reader is encouraged to have a "visual" understanding of these rather abstract concepts.

A point of emphasis in what follows is the idea of *universal properties*: we want to construct certain topological spaces not quite directly but by specifying ways to map either in or out of them. This will lead us to the idea of quotient spaces which is the final topic in this chapter.

## 1. The definition of a topological space

Recall that $\mathcal{P}(\mathrm{X})$ denotes the power set of X, i.e., the set of all subsets of X. A good way to think about the power set is as functions $f : \mathrm{X} \to \{0, 1\}$ which we sometimes denote as $\{0, 1\}^{\mathrm{X}}$. Under this correspondence, a subset $\mathrm{A} \subset \mathrm{X}$ is sent to its characteristic function:

$$\chi_{\mathrm{A}}(x) = \begin{cases} 1 & x \in \mathrm{A} \\ 0 & x \notin \mathrm{A}, \end{cases}$$

and a function $f : \mathrm{X} \to \{0, 1\}$ is sent to its preimage over 1. The notion of a topology selects some elements of $\mathcal{P}(\mathrm{X})$ and declares them to be *open*.

Definition 1.0.1. A **topology** on a set X is a collection $\mathcal{T} \subset \mathcal{P}(\mathrm{X})$ subject to the following properties:

(1) $\emptyset$ and X are in $\mathcal{T}$,
(2) $\mathcal{T}$ is closed under arbitrary unions,
(3) $\mathcal{T}$ is closed under finite intersections.

The pair $(\mathrm{X}, \mathcal{T})$ is called a **topological space**. Elements in $\mathcal{T}$ are called **open**. If $x \in \mathrm{X}$, then any element $\mathrm{U} \subset \mathcal{P}(\mathrm{X})$ such that $x \in \mathrm{U}$ is called a **open neighborhood** of $x$.

The language is meant to evoke certain visual ideas that you should have picked up in analysis. However, this intuition comes with a lot of caveats as we will soon learn throughout the class.

Remark 1.0.2 (Closed sets). A **closed set** $\mathrm{Z} \subset \mathrm{X}$ is defined to be the complement of an open set. In other words: $\mathrm{X} \setminus \mathrm{Z} \in \mathcal{T}$. Rather than specifying the collection of opens, one can also specify the collection of closed sets instead in order to define a topology.

As with any new mathematical definition, the first order of business is to demonstrate that it is an interesting one by coming up with a list of examples.

Example 1.0.3 (Discrete and indiscrete topology). We can always come up with the minimum topology on a space by asking that the opens are only X and $\emptyset$; this is called the **indiscrete topology**. On the opposite end, we have the **discrete topology** where *all* sets are declared to be open, i.e., we set $\mathcal{T} := \mathcal{P}(\mathrm{X})$. One motivation for the name of the latter is the seemingly strange fact that singleton sets, i.e. points, are also open in X.

EXAMPLE 1.0.4 (The cofinite topology). Declare a set $U \subset X$ to be open if either it is empty or if $X \setminus U$ is finite. This defines a topology on X (check this!).

EXAMPLE 1.0.5 (Topologies on finite sets). In [**Mun00**, Ch.2 Ex.1], we have an example of a **finite topological space**, that is, a topological space whose underlying set is finite. While this might be seen as a "toy example" it turns out that one can build important topological spaces out of finite sets. An interesting topology on the two point set is discussed in Example 2.1.4.

EXAMPLE 1.0.6 (The real line and unit interval). Perhaps the easiest non-trivial topological spaces which are "geometrically intuitive" are the unit interval $[0, 1]$ and the real line $\mathbb{R}$ (with its usual topology). Let us recall what it means to be open in $\mathbb{R}$: a set $U \subset \mathbb{R}$ is open if for any $x \in U$ there exists an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset U$. In other words, one can draw a small open interval surrounding $x$ and still stay within U. Now we can induce a topology on $[0, 1]$ by declaring that a set $V \subset [0, 1]$ is open whenever $V = U \cap [0, 1]$ for any open $U \subset \mathbb{R}$. In the unit interval, it is also easy to see that a set is open if and only if it can be written as a union of open intervals $(a, b)$ and half-opens of the form $(a, 1]$ and $[0, b)$ for $0 < a, b < 1$. However, we will see that there are already interesting and puzzling closed subsets of the unit interval.

1.0.7. *The Cantor Set I.* One of the most interesting, yet elementary, topological spaces that one meets is the so-called **Cantor set**. Before we delve into it, let us warm up and recall the following fundamental result due to Cantor:

THEOREM 1.0.8. *There is no bijection between* $\mathbb{N}$ *and* $\mathbb{R}$. *In other words, the real numbers is uncountable.*

PROOF. To say that there is bijection between $\mathbb{N}$ and $\mathbb{R}$ means that there is a $\mathbb{N}$-indexed list of real numbers. So suppose there exists one so that $\mathbb{R} = \{x(n)\}_{n\in\mathbb{N}}$. Let us write the fractional part of $x(n)$ as

$$\mathrm{frac}(x(n)) = 0.x_1(n)x_2(n)\cdots .$$

Choose a sequence of digits $y_1, y_2, \cdots$ such that $y_i \neq x_n(n)$ and consider the real number

$$y := 0.y_1y_2\cdots .$$

Now, by construction, $y$ disagrees with $\mathrm{frac}(x(n))$ from the list and thus could not have been in the list in the first place. □

You will prove this in the homework:

COROLLARY 1.0.9. *The set of all irrational numbers is uncountable.*

The result itself is of remarkable importance in modern mathematics, it states that there are more than one "size" of infinity because two infinite sets can be non-bijective. The method of proof is also remarkable and here's a generalization which clarifies the nature of the proof.

THEOREM 1.0.10. *There is no bijection between a set* X *and* $\mathcal{P}(X)$.

PROOF. If there was, then there will be a map $f : X \to \mathcal{P}(X)$ which is bijective so, in particular, surjective. Consider a subset $B \subset X$ defined to be

$$B := \{x : x \notin f(x)\}$$

Since $f$ is surjective, $B = f(x)$. However, we see that $x \in B$ if and only if $x \notin f(x)$ if and only if $x \notin B$. This is a contradiction. □

We can reprove Theorem 1.0.8 where we instead use a base 2 expansion of the real numbers instead of the base 10. Indeed, Theorem 1.0.10 implies that $\mathbb{N}$ cannot surject onto $\{0, 1\}^{\mathbb{N}}$ which are sequences of binary digits. The existence of binary expansions of real numbers then says that $\mathbb{N}$ cannot surject onto $\mathbb{R}$. We see that Cantor was very fascinated with the idea of diagonalization, expansions of real numbers and so on. It led him to the construction of his eponymous set.

Construction 1.0.11 (The Cantor Set). Begin with $A_0 := [0, 1]$ the unit interval. We set $A_1 \subset A_0$ to be the subset obtained by deleting its middle third:

$$A_1 := A_0 \smallsetminus (1/3, 2/3).$$

Then we set $A_2$ to be obtained from $A_1$ by deleting two segments which are middle thirds:

$$A_2 := A_1 \smallsetminus ((1/9, 2/9) \cup (7/9, 8/9)).$$

Proceed inductively this way

$$A_n := A_{n-1} \smallsetminus \bigcup_{k=0}^{\infty} (1 + 3k/3^n, 2 + 3k/3^n).$$

We set

$$\mathcal{C} := \bigcap_{n>0}^{\infty} A_n.$$

It is topologized by saying that $U \subset \mathcal{C}$ is open whenever $U = V \cap \mathcal{C}$ for any open $V \subset \mathbb{R}$.

By construction, we see that a number $x \in [0, 1]$ is in the Cantor set if and only if there exists a ternary expansion of $x$ without the appearance of the digit 1. To see this, imagine that I have an element of $[0, 1]$ and I want to know if it survive each "cut" in the sequence. If it survives the first step, then it either moves "left" or "right" — if it moves left then one can use 0 in the ternary expansion and 2 if it moves right. This goes on ad infinitum: you can trace each element in the Cantor set through an infinite binary tree through each cut.

Now, we need to topologize the Cantor set. Since $\mathcal{C} \subset [0, 1]$ we can simply inherit the topology from the $[0, 1]$: an set $U \subset \mathcal{C}$ is open if and only if $U = \mathcal{C} \cap V$ where $V \subset [0, 1]$ is open. The following are important properties of the Cantor set, also to be proved in the homework exercises:

Lemma 1.0.12 (Basic properties of the Cantor set). *The following holds of $\mathcal{C}$:*

*(1) it is a closed subset of $[0, 1]$;*
*(2) it is an uncountable set;*
*(3) it contains no intervals: there is no set of the form $[a, b]$ where $a < b$ contained in $\mathcal{C}$.*

By Corollary 1.0.9 we see that irrationals are uncountable; the ones in $[0, 1]$ are also easily seen to be uncountable. However, as we will soon see, the irrationals are not closed in $[0, 1]$ unlike the Cantor set. In fact, the irrationals are in a technical sense, "dense" within each closed interval in $[0, 1]$ but this fails drastically for the Cantor set. This is a kind of "smallness" (technical term: compactness) condition that the Cantor set enjoys.

**1.1. Topology via bases.** We are still in the theme of how to write down topologies on a set. Often, one wants to specify a topology using only a partial collection of open sets. We have already seen this in real analysis; we begin by reviewing the idea of open balls in a slightly more abstract context.

Example 1.1.1 (Metric spaces). In your analysis course, you might have learned the idea/definition of a metric space — perhaps at least on subsets of $\mathbb{R}^n$. Intuitively, a metric specifies a way to measure distances between points in a set. We recall that a **metric space** is a space X equipped with a function

$$d : X \times X \to \mathbb{R}$$

such that

(1) $d(x, y) \geqslant 0$ for any $x, y \in X$
(2) $d(x, x) = 0$
(3) $d(x, y) = d(y, x)$
(4) we have the triangle inequality

$$d(x, z) \leqslant d(x, y) + d(y, z).$$

Recall that we have the notion of **$\epsilon$-balls** for $\epsilon > 0$, i.e.,

$$\mathrm{B}(x_0, \epsilon) := \{y : d(x_0, y) < \epsilon\}.$$

These are the basic building blocks for arguments in real analysis and are the building blocks for opens in X. The notion of a basis makes this precise.

DEFINITION 1.1.2. Let X be a set. A collection $\mathcal{B} \subset \mathcal{P}(\mathrm{X})$ is a **basis** for a topology if:

(1) for each $x \in \mathrm{X}$, there is at least one basis element that contains $x$;
(2) if $x$ is in the intersection of two basis elements $\mathrm{B}_1 \cap \mathrm{B}_2$, then there is a basis element $\mathrm{B}_3$ with $x \in \mathrm{B}_3 \subset \mathrm{B}_1 \cap \mathrm{B}_2$.

The **topology generated by** $\mathcal{B}$ is the topology defined as follows: an set $\mathrm{U} \subset \mathrm{X}$ is open if for each $x \in \mathrm{U}$ there exists an element $\mathrm{B} \in \mathcal{B}$ such that $x \in \mathrm{B} \subset \mathrm{U}$. The following is a routine check and left to the reader.

LEMMA 1.1.3. *The topology generated by a basis $\mathcal{B}$ is indeed a topology. In fact, opens are exactly those which can be expressed as an arbitrary unions of elements in $\mathcal{B}$.*

EXAMPLE 1.1.4 (The Euclidean topology on $\mathbb{R}^n$). Continuing the example of metric spaces, we see that the collection $\{\mathrm{B}(x_0, \epsilon)\}_{x_0 \in \mathrm{X}}$ forms a basis for the topology on X; the second condition is most easily verified by drawing a picture. By Lemma 1.1.3 we see that a set $\mathrm{U} \subset \mathrm{X}$ is open if and only if for each $x \in \mathrm{U}$ there exists a $\epsilon > 0$ such that $\mathrm{B}(x_0, \epsilon) \subset \mathrm{U}$. We will use the notation $(\mathbb{R}^n, \mathrm{Euc})$ to denote $\mathbb{R}^n$ with its usual metric topology; here Euc is meant to indicate the Euclidean topology which is the usual name for this topology.

EXAMPLE 1.1.5 (Metrics and norms). We are quite familiar with metrics from analysis or just from everyday experience. Here is a collection of "unusual" metrics on $\mathbb{R}^n$. Let $p \geqslant 1$ be an integer. Then the **$\ell^p$-norm on** $\mathbb{R}^n$ is defined as the quantity

$$||x = (x_1, \cdots, x_n)||_p = (|x_1|^p + |x_2|^p + \cdots |x_n|^p)^{\frac{1}{p}}, x \in \mathbb{R}^n.$$

Then the **$\ell^p$-metric** is given by

$$d_p(x, y) := ||x - y||_p$$

and the reader is encourages to check that this indeed a metric. We remark that when $p = 2$ we recover the usual Euclidean metric that we are familiar with. When $p = 1$ we get that $d_1(x, y) = \sum_{i=1}^n |x_i - y_i|$, sometimes called the **Manhattan metric**. We also have the $p = \infty$ norm which is declared to be

$$||x||_\infty = \max\{|x_i|\}$$

and the metric

$$d_\infty(x, y) = \max\{|x_i - y_i|\}.$$

The reader is encourages to draw the unit ball, centered at zero, for these various norms.

REMARK 1.1.6 (Closeness of points). At this point we have made a connection with analysis where a lot of emphasis was put on the notion of points being close to each other. Intuitively, a sequence $\{x_n\}$ (of, say, real numbers or elements of $\mathbb{R}^n$) converges to $x$ (in which case $x$ is a **limit point**) if $x$ can be approximated up to an arbitrary degree of precision using the $x_n$'s. Hence we can speak of elements of $x_n$ approaching $x$ as close as possible. More precisely, we say that a sequence $\{x_n\}$ converges to $x$ if and only if for any $\epsilon > 0$ there exists an $\mathrm{N} \gg 0$ such that for all $n \geqslant \mathrm{N}$, we have that $x_n \in \mathrm{B}(x, \epsilon)$. This already invokes the notion of a basis of a topology; we could instead have used a more flexible notion: for each open set U containing $x$, there exists an $\mathrm{N} \gg 0$ such that $x_n \in \mathrm{U}$ for $n \geqslant \mathrm{N}$. In fact, we can formulate the latter notion in *any* topological space. Hence one way to think about a topology, or rather an open set, is a measure of "closeness of points."

Given a topology, I encourage you to think in these terms (which may or may not be useful). What would "analysis" or "convergence" look like?

**1.2. Varying topologies on a set.** There are usually more than one topologies on a set X. If $\mathcal{T} \subset \mathcal{T}'$ or $\mathcal{T}' \subset \mathcal{T}$ we say that they are **comparable** and if we are in the first situation, we say that $\mathcal{T}$ is **coarser** than $\mathcal{T}'$ or that $\mathcal{T}'$ is **finer** than $\mathcal{T}$. We should always think that a coarser topology has *less* open sets than a finer one.

EXAMPLE 1.2.1. The discrete topology is always finer than every other topology, while the indiscrete topology is always coarser than every other topology.

Bases somewhat clarifies what is going on.

LEMMA 1.2.2. *Let $\mathcal{B}, \mathcal{B}'$ be bases generating topologies $\mathcal{T}, \mathcal{T}'$ respectively. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if for each $x \in \mathrm{X}$ and each basis $\mathrm{B} \in \mathcal{B}$ containing $x$, there exists $\mathrm{B}' \in \mathcal{B}'$ such that $x \in \mathrm{B}' \subset \mathrm{B}$.*

PROOF. The "only if" direction is easy and left to the reader. Now, if $\mathrm{U} \in \mathcal{T}$ we wish to prove that $\mathrm{U} \in \mathcal{T}'$. To do so, we need to show that for each $x \in \mathrm{U}$ there exists a $\mathrm{B}'$ such that $x \in \mathrm{B}' \subset \mathrm{U}$. But now, by Lemma 1.1.3, there exists a $\mathrm{B} \in \mathcal{B}$ such that $x \in \mathrm{B} \subset \mathrm{U}$ but, by hypothesis, we further have a $\mathrm{B}' \in \mathcal{B}'$ such that $x \in \mathrm{B}' \subset \mathrm{B}$. □

EXAMPLE 1.2.3 (Lower limit topology). The **lower limit topology** on $\mathbb{R}$ is the topology whose basis is given by intervals of the form $[a, b)$ where $b > a$; we leave it to the reader to verify that this is indeed a basis. The lower limit topology is finer than the usual topology on $\mathbb{R}$ given by its usual metric where $(a, b)$ forms a basis. Indeed, using Lemma 1.2.2, it suffices to note that for each $x \in (a, b)$, the interval $[x, b)$ is contained in $(a, b)$.

However, the lower limit topology is somewhat exotic: $[a, b)$ is not generally open in $\mathbb{R}$ so this topology is strictly finer. What does analysis look like in the lower limit topology? In fact, convergence is actually harder: a sequence $\{x_n\}$ converges to $x$ if and only if it converges to $x$ in the conventional sense and $x_n \geqslant x$ for all $n \gg 0$. In this topology, points can only be approached from above; it is also sometimes called the **uphill topology**. In fact, the lower limit topology is *not metrizable* in that it does not come from a metric (we will show this later).

We will soon reformulate this idea when we have learned the notion of a continuous function of topological spaces. In any case, we have seen that there are many possible topologies that one can put on a set. We examine this in the case of the (open) box.

1.2.4. *Topologies on $\mathbb{R}^2$ and orderings.* The Euclidean topology on the real line is an example of a metric topology; it is also an example of an *order topology*, a general concept we will soon encounter. Roughly speaking, this is a topology on a set with a notion of inequality, whose bases are given by intervals. In this subsection, we will discuss the order topology on $\mathbb{R}^2$ and on the "box" $\mathrm{I} \times \mathrm{I}$.

EXAMPLE 1.2.5 (The order topology on $\mathbb{R}^2$). There is an ordering on $\mathbb{R}^2$ given as follows:

$$a \times b < c \times d$$

if $a < c$ or $a = c$ and $b < d$. It is instructive to draw the intervals defining this order relation in the plane. The order topology on $\mathbb{R}^2$ is the topology generated by the basis:

$$\mathcal{B} = \{(a \times b, c \times d) : a < c, \text{ or } a = c \text{ and } b < d\}.$$

This topology is actually finer than the usual Euclidean topology in $\mathbb{R}^2$.

The next example introduces us naturally to the abstract notion of an order topology as well as the procedure of restricting topologies.

EXAMPLE 1.2.6 (Topologies on $\mathrm{I} \times \mathrm{I}$). Consider the interval $\mathrm{I} := [0, 1] \subset \mathbb{R}$. Then we can restrict the order topology on $\mathbb{R}$; in fact we can just restrict a basis given by the collection of open intervals $\{(a, b) : a < b\}$. This specifies the following collection

$$(a, b), (a, 1], [0, b), \emptyset, [0, 1]$$

which forms a basis for topology on $[0,1]$. This is exactly the topology discussed in Example 1.0.6.

In two dimensions, we have the "box": $\mathrm{I}\times\mathrm{I}=\{(a,b):0\leqslant a\leqslant 1, 0\leqslant b\leqslant 1\}$. There are at least two ways to give a topology in $\mathrm{I}\times\mathrm{I}$ involving orders; we first note that orderings on a set give a natural topology, abstracting the previous examples:

Definition 1.2.7. The **order topology** on a totally ordered set X with more than one element is the topology generated by the basis $\mathcal{B}$ consisting of

(1) all the open intervals $(a,b)$;
(2) $[a_0,b)$ if $a_0$ is the smallest element of X (if it exists);
(3) $(a,b_0]$ if $b_0$ is the largest element of X (if it exists).

Example 1.2.5 is the order topology associated to the natural lexicographic/dictionary ordering on $\mathbb{R}^2$. We can also order $\mathrm{I}\times\mathrm{I}$ with the lexicographic ordering which is just the restriction of the ordering on $\mathbb{R}^2$. The other option is to give it the subspace topology

Definition 1.2.8. If $\mathrm{A}\subset\mathrm{X}$ is a subset of a set X and $\mathcal{T}$ is a topology on A, then the **subspace** topology is the topology on A given by

$$\mathcal{T}_{\mathrm{A}}:=\{\mathrm{U}\cap\mathrm{A}:\mathrm{U}\in\mathcal{T}\}.$$

It is a routine check that this defines a topology on A.

One thing to keep in mind about subspaces is that if $\mathrm{U}\subset\mathrm{A}$ is an open, then it may or may not be open in the bigger space X. For example, $\mathrm{I}\subset\mathbb{R}^2$ is clearly not open, but $\mathrm{I}\subset\mathrm{I}$ is open. Of course if $\mathrm{A}\subset\mathrm{X}$ is actually open, then if $\mathrm{U}\subset\mathrm{A}$ is open, it must be open in X.

Example 1.2.9 (Two topologies on $\mathrm{I}\times\mathrm{I}$). It should somewhat disturb you that the subspace topology on $\mathrm{I}\times\mathrm{I}$ inherited from the order topology on $\mathbb{R}^2$ is strictly finer than the order topology. Note that given a basis of the order topology on $\mathrm{I}\times\mathrm{I}$, we can simply extend it to one on $\mathbb{R}^2$. However, subsets of the form $\{a\}\times(a,1]$, $0<a<1$ are open in the subspace topology but not open in the order topology.

## 2. Continuous functions

In modern mathematics, objects are never defined without context. This means that we want to define the correct notion of functions (the better name is morphisms) between them. In the context of topology, the correct notion of morphism is not a mere function of underlying sets but one that also preserves opens.

### 2.1. Continuous functions and homeomorphisms.

Definition 2.1.1. A **continuous function/continuous map/morphism** between topological spaces

$$f:(\mathrm{X},\mathcal{T})\to(\mathrm{Y},\mathcal{T}')$$

is a function $f:\mathrm{X}\to\mathrm{Y}$ such that for any $\mathrm{U}\in\mathcal{T}'$, $f^{-1}(\mathrm{U})\in\mathcal{T}$.

When the context is clear, we will simply write a continuous function as $f:\mathrm{X}\to\mathrm{Y}$.

Remark 2.1.2 (Continuous functions and coarser/finer topologies). Consider the identity map $\mathrm{X}\to\mathrm{X}$, then we can ask when $(\mathrm{X},\mathcal{T})\to(\mathrm{X},\mathcal{T}')$ induces a continuous function. This is the same thing as asking that any $\mathrm{U}\in\mathcal{T}'$ is also in $\mathcal{T}$ which means that the identity is a continuous function if and only if $\mathcal{T}$ is finer than $\mathcal{T}'$. This is a nice way to think of a continuous map: roughly a map of topological spaces $f:\mathrm{X}\to\mathrm{Y}$ is a "*refinement*" of Y by X.

Therefore, $(\mathrm{X},\delta)\xrightarrow{\mathrm{id}}(\mathrm{X},\mathcal{T})$ is always continuous where $\delta$ is the discrete topology and $(\mathrm{X},\mathcal{T})\xrightarrow{\mathrm{id}}(\mathrm{X},\delta')$ is always continuous where $\delta'$ is the indiscrete topology. On the other hand, $(\mathbb{R},\mathrm{Euc})\xrightarrow{\mathrm{id}}(\mathbb{R},\text{lower limit})$ is evidently not continuous but it is if we swap the order of the topologies.

Remark 2.1.3. Because topologies can also be defined via closed sets, it is easy to see that a function $f : \mathrm{X} \to \mathrm{Y}$ is continuous if and only if for any closed subset $\mathrm{B} \subset \mathrm{Y}$, $f^{-1}(\mathrm{B}) \subset \mathrm{X}$ is closed. We leave the details to the reader.

We will often use the following continuity criterion:

Lemma 2.1.4. *Let $f : \mathrm{X} \to \mathrm{Y}$ be a function between topological spaces. Then the following are equivalent:*

*(1) $f$ is a continuous function;*
*(2) Local criterion: for each $x \in \mathrm{X}$ and each open neighborhood $f(x) \in \mathrm{V} \subset \mathrm{Y}$, there exists an open neighborhood $x \in \mathrm{U}$ such that $f(\mathrm{U}) \subset \mathrm{V}$.*

Proof. If $f$ is continuous, and $f(x) \subset \mathrm{V} \subset \mathrm{X}$, then $f^{-1}(\mathrm{V})$ is automatically open by definition. On the other hand, if $\mathrm{V} \subset \mathrm{Y}$, then we can write $f^{-1}(\mathrm{V}) = \cup_{x \in f^{-1}(\mathrm{V})} \mathrm{U}_x$ where $\mathrm{U}_x$ is the open neighborhood produced by hypothesis. ☐

The next definition is the correct notion for two spaces to be isomorphic.

Definition 2.1.5. Let $f : \mathrm{X} \to \mathrm{Y}$ be a bijection of sets so that we have an inverse $g : \mathrm{Y} \to \mathrm{X}$. Suppose that $f$ is continuous, then we say that it is a **homeomorphism** if $g$ is also continuous. We shall write $\mathrm{X} \cong \mathrm{Y}$ if X is homeomorphic to Y.

Remark 2.1.6 (Inverses are unique). If $f : \mathrm{X} \to \mathrm{Y}$ is a bijection of sets, then its inverse is unique. Therefore, to check that a continuous bijection $f$ is homeomorphism, it suffices to check that its inverse function $g : \mathrm{Y} \to \mathrm{X}$ is continuous.

Remark 2.1.7 (Open maps). A homeomorphism is an example of an **open map**: a continuous function $f : \mathrm{X} \to \mathrm{Y}$ such that for any open set $\mathrm{U} \subset \mathrm{X}$, $f(\mathrm{U})$ is open. Indeed, if $f$ is a homeomorphism with inverse $g$, then $f(\mathrm{U}) = g^{-1}(g(f(\mathrm{U}))) = g^{-1}(\mathrm{U})$ which is open since $g$ is continuous.

Example 2.1.8 (Bijections which are not homeomorphisms). The identity map is often continuous map without being bijection; $(\mathbb{R}, \text{lower limit}) \to (\mathbb{R}, \text{Euc})$ is continuous but its inverse is $(\mathbb{R}, \text{Euc}) \xrightarrow{\text{id}} (\mathbb{R}, \text{lower limit})$ which is not continuous.

There are two general ways in which continuous bijections can fail to be homeomorphisms:

(1) The domain fails to be compact: define:
$$f : [0, 1) \to \mathrm{S}^1 \qquad t \mapsto (\cos 2\pi t, \sin 2\pi t).$$
One can easily check that this is a continuous bijection: it gives a way to parametrize the unit circle using trigonometric functions. However, the function $f$ is not an open map because the image of half-open interval $[0, a)$ for $a > 0$ is not open in $\mathrm{S}^1$.
(2) The target fails to be Hausdorff: we have yet to define the notion of a Hausdorff space. Roughly it means that any two points can be "divided" by open sets. For example, the unit interval with the indiscrete topology is *not* Hausdorff since the only two opens are the entire set and $\emptyset$. Since the Euclidean topology is strictly finer than the indiscrete topology, we see that we have a continuous bijection which is not a homeomorphism.

Example 2.1.9 (A homeomorphism between $(-1, 1)$ and $\mathbb{R}$). There are many ways to define homeomorphisms between $(-1, 1)$ and $\mathbb{R}$. Here is one: note from high school trigonometry that the functions
$$\tan : (-\pi/2, \pi/2) \to \mathbb{R} \qquad \arctan : \mathbb{R} \to (-\pi/2, \pi/2)$$
are bijections. It is then a basic fact that these trigonometric functions are continuous. We can also check that the scaling function $(-1, 1) \to (-\pi/2, \pi/2)$ is continuous. Hence composing both functions lead us to the desired conclusion.

The following lemma records elementary properties about continuous functions which we will use throughout this course; we will use the following notation: if $\mathrm{A} \subset \mathrm{X}$ is a subset, and $f : \mathrm{X} \to \mathrm{Y}$ is a function we write $f|_{\mathrm{A}} : \mathrm{A} \to \mathrm{X}$ for the restriction of $f$ to A.

Lemma 2.1.10. *The following properties hold for spaces* $X, Y, Z$*:*

*(1) constant functions are continuous;*
*(2) inclusion of subspaces* $j : A \to X$ *is continuous;*
*(3) continuous functions are closed under compositions: if* $f : X \to Y, g : Y \to Z$ *are continuous functions, then* $g \circ f : X \to Z$ *is;*
*(4) continuity is local: if* $X = \cup U_\alpha$ *where* $U_\alpha$ *is an open of* $X$*, then* $f : X \to Y$ *is continuous if and only if* $f|_{U_\alpha} : U_\alpha \to Y$ *is.*
*(5) continuous functions is a factorization system: every continuous map* $f : X \to Y$ *factors uniquely as*

$$X \xrightarrow{g} f(X) \xrightarrow{h} Y$$

*where* $g, h$ *are continuous functions and* $g$ *is a surjection and* $h$ *is an injection.*
*(6) Gluing: assume* $A, B$ *are closed subsets of* $X$ *such that* $X = A \cup B$*. If* $f : A \to Y, g : B \to Y$ *are continuous functions such that* $f|_{A \cap B} = g|_{A \cap B}$*, then there is a unique continuous function* $h : X \to Y$ *such that* $h|_A = f, h|_B = g$*. The result also holds if* $A, B$ *are both open.*

Proof. See [**Mun00**, Theorem 18.2-3]. Let us briefly discuss the last point. We let $h : X \to Y$ be $f$ on $A$ and $g$ on $B$; since the values agree on $A \cap B$, we see that $h$ is well-defined. We then have to check that $h$ is continuous. Let $Z \subset Y$ be a closed subset; it suffices to prove that $h^{-1}(Z)$ is closed in $X$. But $h^{-1}(Z) = f^{-1}(Z) \cup g^{-1}(Z)$; since $f$ and $g$ are continuous both components of the union are closed and therefore $h^{-1}(Z)$ is closed.

□

The gluing lemma of Lemma 2.1.10 is one important way that we can construct continuous functions out of new ones. This theme will be explored further as the class progresses.

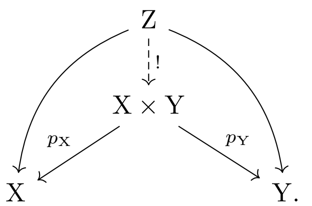
## 3. Products

I want to use a little bit of the language of category theory (which we secretly have already experienced). If $X$ and $Y$ are two sets, we can consider its cartesian product $X \times Y$ which, in coordinates, we write as pairs $(x, y)$. We sort of know what this is and have experienced it (e.g. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$) but how do we know what really is? Here's a precise statement of what the product really "is":

Lemma 3.0.1. *Let* $X, Y$ *be sets, and let* $Z \to X, Z \to Y$ *be maps. Then there exists a unique map* $Z \to X \times Y$ *that commutes with the projections* $X \leftarrow X \times Y \to Y$*.*

In category theory, one experiences mathematical statements via diagrams. The statement of Lemma 3.0.1 can be summarized in the following diagram:

(3.0.2)




The "!" is meant to indicate uniqueness. It might not really be evident that this is a powerful viewpoint in the world of sets: it is so easy to understand cartesian products and jive it with our experience. However, we will see that this can be very useful already in basic topology.

Remark 3.0.3. Let us formulate uniqueness more precisely. For any two sets $X, Y$ let us write $\mathrm{Hom}(X, Y)$ for the set of continuous maps between $X$ and $Y$. Then the uniqueness assertion of Lemma 3.0.1 is the following: precomposition gives

$$\mathrm{Hom}(Z, X \times Y) \to \mathrm{Hom}(Z, X) \times \mathrm{Hom}(Z, Y).$$

Then Lemma 3.0.1 can be reformulated as saying that the above map is a bijection.

**3.1. Products in topology.** The question that we wish to answer is:

QUESTION 3.1.1. Given a collection of topological spaces $\{X_\alpha\}_{\alpha \in I}$, what is the "correct" topology to put on $\prod_{\alpha \in A} X_\alpha$?

When I is a finite set, the answer is quite easy; we contend ourselves with the case of $|I| = 2$ and proceed via induction.

DEFINITION 3.1.2. Let $X, Y$ be topological spaces. Then the **product topology** on $X \times Y$ is the topology generated by bases

$$\mathcal{B} := \{U \times V : U \text{ open in } X, V \text{ open in } Y\}.$$

EXAMPLE 3.1.3 (The Euclidean topology). The topology from Example 1.1.4 is the product topology given by the $n$-fold product of $\mathbb{R}^1$ (with the order/Euclidean topology). The product topology provides, as basis, products of intervals $(a_1, b_1) \times (a_2, b_2) \times \cdots (a_n, b_n)$ which look more like boxes instead of balls. However, it is easy to see that one can refine boxes by balls and vice versa.

EXAMPLE 3.1.4 (The lexicographic topology revisited). Let $\mathbb{R}_\delta$ be $\mathbb{R}$ with the discrete topology. The product topology on $\mathbb{R}_\delta \times \mathbb{R}$ is the same as the lexicographic topology on $\mathbb{R}^2$. The point here is that a basis for both topologies are given by intervals of the form $\{a\} \times (c, d)$ as one can verify.

We note the following observation: the product topology renders the maps $p_X$ and $p_Y$ automatically continuous. Indeed, $p_X^{-1}(U) = U \times Y$ which is a basis element for the product topology; similarly for $p_Y$. Hence, if we endow $X \times Y$ with the product topology we are allowed to contemplate the diagram (3.0.2) within the world of topological spaces. This gives a candidate answer to Question 3.1.1: we can just try to upgrade Lemma 3.0.1 to the world of topological spaces.

PROPOSITION 3.1.5 (Universal property of finite products). *Let* $X, Y$ *be spaces. Then* $X \times Y$ *with the product topology is the unique topological space (up to homeomorphism) satisfying the following property: given continuous functions* $Z \to X, Z \to Y$*, then there exists a unique continuous function* $Z \to X \times Y$ *commuting with the projections* $X \leftarrow X \times Y \to Y$.

PROOF. Suppose that we are given continuous function $g : Z \to X, h : Z \to Y$. Then, by Lemma 3.0.1 we obtain a unique function $f : Z \to X \times Y$ rendering the diagram (3.0.2) commutative. To prove the proposition, we need only show that $f$ is, in fact, a continuous function. It suffices to check that $f^{-1}(U \times V)$ is open for $U \subset X, V \subset Y$ opens. Since (3.0.2) commutes as a diagram of sets, $f^{-1}(U \times V) = h^{-1}(U) \cap g^{-1}(V) \subset Z$. By assumption, $h^{-1}(U), g^{-1}(V)$ are opens and thus we are done.

Now, we prove that if W is any other space satisfying the same universal property then $W \cong X \times Y$. Universal properties then furnishes us unique *continuous* functions $f : W \to X \times Y$ and $g : X \times Y \to W$ which are mutual inverses (because they are in the world of sets). □

Proposition 3.1.8 is not mere philosophical curiosity. It tells us that we can easily construct maps into products. The universal property further shines in the world of infinite products where there are two competing topologies one can put on $\prod_\alpha X_\alpha$.

REMARK 3.1.6. While Proposition 3.1.8 characterizes the space $X \times Y$ up to homeomorphism we still need to explicitly construct it as a space, i.e., it does not render Definition 3.1.2 entirely useless. There are, however, abstract reasons for the existence of the product topology once we know more category theory.

DEFINITION 3.1.7. The **product topology** on $\prod_\alpha X_\alpha$ is the topology generated by a basis $\{\mathcal{U} := \prod_{\alpha \in A} U_\alpha\}$ where $U_\alpha \subset X_\alpha$ is open and the terms in $\mathcal{U}$ are given by $X_\alpha$ for all but finitely many $U_\alpha$'s. The **box topology** is the topology generated by a basis $\{\mathcal{U} := \prod_{\alpha \in A} U_\alpha\}$ where $U_\alpha \subset X_\alpha$ is open.

Clearly, the box topology is finer than the product topology and both are extensions of the product topology on finite products.

PROPOSITION 3.1.8 (Universal property of infinite products). *Let* $\{X_\alpha\}_{\alpha\in A}$ *be collection of spaces. Then* $\prod_\alpha X_\alpha$ *with the product topology is the unique topological space (up to homeomorphism) satisfying the following property: to give a continuous function* $Z \to \prod_\alpha X_\alpha$ *commuting with the projections* $\prod_\alpha X_\alpha \to X_\alpha$ *is equivalent to giving maps* $Z \to X_\alpha$ *for each* $\alpha \in A$.

PROOF. Given a collection of continuous functions $f_\alpha : Z \to X_\alpha$, we get a function $f : Z \to \prod X_\alpha$ by universal properties in sets. Here is the key calculation to prove that $f$ is continuous: let $\mathcal{U} = \prod V_\alpha \subset \prod X_\alpha$ be a subset then

$$f^{-1}(\mathcal{U}) = \bigcap_\alpha f_\alpha^{-1}(V_\alpha). \tag{3.1.9}$$

Therefore, we have no guarantee that $f$ is continuous (takes opens to opens) unless the intersection is finite which forces us to set $\mathcal{U}$ to only have finitely many non $X_\alpha$-terms. We leave the details of the proof to the reader. □

Indeed, there is no reason in general that $f : Z \to \prod X_\alpha$ will factor through the map $(\prod X_\alpha, \text{box}) \to \prod X_\alpha$ as the next example illustrates.

REMARK 3.1.10. Consider $\mathbb{R}^\omega = \prod_{\mathbb{N}} \mathbb{R}$, the countable product of copies of $\mathbb{R}$. We have a function $f := (t, t, t, \cdots) : \mathbb{R} \to \mathbb{R}^\omega$; composition with projections $f_n : \mathbb{R} \to \mathbb{R}^\omega \to \mathbb{R}$ are clearly continuous but we claim that $f$ itself is not. Indeed consider the following product of opens

$$U := \prod_{n\geqslant 1}(-\tfrac{1}{n}, \tfrac{1}{n}).$$

By definition, U is open in the box topology. However, $f^{-1}(U)$ is not open in $\mathbb{R}$. For if it does, then it will contain an open interval around 0, i.e., $(-\delta, \delta) \subset \mathbb{R}$. But if this were the case then, $f((-\delta, \delta)) \subset U$ and taking projection we will get that $(-\delta, \delta) \subset (-\frac{1}{n}, \frac{1}{n})$ for all $n$ which is clearly not true!

**3.2. Hilbert cubes I.** The **Hilbert cube** is the space

$$Q := \prod_{n\in\mathbb{N}_{>0}} [0, 1/n],$$

endowed with the product topology. It is a massively interesting space and is the building block for a theory of "infinite dimensional manifolds." Here is a sense in which this space is well-behaved: the product topology has a nice description in terms of a metric.

Consider the set $\mathbb{R}^\omega$, countably many copies of $\mathbb{R}$. There is a nice metric that one can endow on this set: first we set the following metric on $\mathbb{R}$:

$$\overline{d}(x, y) = \min\{|x - y|, 1\}.$$

Then the **uniform metric** on $\mathbb{R}^\omega$ is given by

$$d_\infty((x_i), (y_i)) := \sup\{\overline{d}(x_i, y_i)\}.$$

This is a modification of the sup norm on $\mathbb{R}^n$ that ensures that the metric is well defined on $\mathbb{R}^\omega$. For finite dimensional euclidean spaces, the sup norm and the Euclidean norm induce the same topology. We can ask if this is still true in this infinite-dimensional setting. Evidently, the question that does not make sense because the sum $\sum |x_i|^2$ need not converge. Let X be the subspace of $\mathbb{R}^\omega$ given by those points such that $\sum |x_i|^2 < \infty$. Then:

LEMMA 3.2.1. *The topology induced by the uniform metric is coarser than the topology induced by the Euclidean metric on* X. *Furthermore the topology induced by the Euclidean metric is coraser than the topology induced by the product topology.*

This lemma will be in the homework. Nonetheless, for the Hilbert cube we have:

LEMMA 3.2.2. *On the Hilbert cube, the Euclidean, uniform and product topologies coincide.*

## 4. Quotients

We now come to a completely new way to construct spaces, one that is somehow "intuitive" but not quite easy to formalize. As motivation, let us consider the following example:

EXAMPLE 4.0.1 (Solution sets and tori). Consider the circle of radius one. This space is quite easy to imagine but to give it a topology one might try to write it down as a subspace of $\mathbb{R}^2$. Indeed, we can look at the subset

$$\mathbb{T} = \{(x, y) : x^2 + y^2 = 1\} \subset \mathbb{R}^2.$$

As it is a subset of $\mathbb{R}^2$ we can give it the subspace topology inherited from the Euclidean topology in $\mathbb{R}^2$. This is a perfectly good way to describe the circle.

A generalization of $\mathbb{T}$ is called the **2-torus**

$$\mathbb{T}^2 = \mathbb{T} \times \mathbb{T}.$$

We give it the product topology and visualize it as a donut. This is also a perfectly good way to construct the 2-torus. There is even a way to write $\mathbb{T}^2$ as embedded in $\mathbb{R}^3$:

$$\mathbb{T}^2 \cong \{(x, y, z) : (2 - \sqrt{x^2 + y^2})^2 + z^2 = 1\}.$$

In these two ways of describing the 2-torus, it comes with an embedding in $\mathbb{R}^3$ which makes the space quite concrete as it comes with a natural set of coordinates. However, in algebraic topology, one wants to describe the 2-torus in a way which is independent of any embedding. After all, we do not really care about finer information on it. To do so, we can present it as a certain quotient space.

Our goal is to explain the following theorem:

THEOREM 4.0.2 (Quotients). *Let* $\mathrm{X}$ *be a topological space and let* $\sim$ *be an equivalence relation on the underlying set of* $\mathrm{X}$. *Then there exists a topology on* $\mathrm{X}/\sim$ *characterized by the following universal property:*

*(1) the quotient map* $q : \mathrm{X} \to \mathrm{X}/\sim$ *is continuous;*
*(2) let* $\mathrm{Z}$ *be a topological space and* $f : \mathrm{X} \to \mathrm{Z}$ *be a continuous function such that whenever* $x \sim x'$ *then* $f(x) = f(x')$, *then there exists a unique continuous function* $\mathrm{X}/\sim \to \mathrm{Z}$ *rendering the following diagram commutative:*

$$\begin{array}{ccc} \mathrm{X} & \xrightarrow{f} & \mathrm{Z} \\ \big\downarrow q & \nearrow & \\ \mathrm{X}/\sim & & \end{array}$$

*(3)* $q : \mathrm{X} \to \mathrm{X}/\sim$ *is unique in the sense that if* $q' : \mathrm{X} \to \mathrm{Y}$ *is any other continuous function satisfying property (2), then there is a homeomorphism* $\varphi : \mathrm{X}/\sim \to \mathrm{Y}$ under $\mathrm{X}$ *in the sense that the diagram commutes:*

$$\begin{array}{ccc} \mathrm{X} & \xrightarrow{f} & \mathrm{Y} \\ \big\downarrow q & \nearrow \varphi & \\ \mathrm{X}/\sim & & \end{array}$$

PROOF. We first construct a candidate topology on $\mathrm{X}/\sim$. Note that the map $q : \mathrm{X} \to \mathrm{X}/\sim$ is surjective; we declare that $\mathrm{U} \subset \mathrm{X}/\sim$ is open if and only if $q^{-1}(\mathrm{U}) \subset \mathrm{X}$ is open. To check that this is indeed a topology, we use the standard equalities from set theory

$$q^{-1}(\bigcup \mathrm{U}_\alpha) = \bigcup q^{-1}(\mathrm{U}_\alpha) \qquad q^{-1}(\cap_{i=1}^{n} \mathrm{U}_i) = \cap_{i=1}^{n} q^{-1}(\mathrm{U}_i).$$

Unwinding definitions, we note that an open set of $\mathrm{X}/\sim$ can be described to consist of those equivalence classes whose union is itself an open set in $\mathrm{X}$. By construction $q$ is clearly a continuous map.

We now prove (2). First, the condition on $f$ is exactly the same thing as saying that $f|_{q^{-1}(x)} : q^{-1}(x) \to \mathrm{Z}$ is constant for all $x \in \mathrm{X}$ since the preimages of $q$ are exactly the equivalence classes of $\sim$. Therefore, the map of sets $\widetilde{f} : \mathrm{X}/\sim \to \mathrm{Z}$ is uniquely determined by the requirement that it sends the class of $x$ to $f(x)$. It suffices to prove that $\widetilde{f}$ is is open. If $\mathrm{U} \subset \mathrm{Z}$ is an open, then to check that $\widetilde{f}^{-1}(\mathrm{U})$ is open it suffices to check it after taking further inverse image along $q$ (by construction). But this is exactly $f^{-1}(\mathrm{U}) \subset \mathrm{X}$ which is open since $f$ is continuous.

It is now easy to check (3) because the universal properties furnish mutually inverse functions which are both continuous. □

REMARK 4.0.3. Again, we can sharpen the uniqueness assertion of Theorem 4.0.2(2) by formulating it this way. Let $\mathrm{Hom}(\mathrm{X}, \mathrm{Y})$ be the set of continuous functions between two spaces X and Y. Then, by precomposition, we have a map of sets

$$\mathrm{Hom}(\mathrm{X}/\sim, \mathrm{Y}) \to \mathrm{Hom}(\mathrm{X}, \mathrm{Y}) \qquad f \mapsto f \circ q.$$

Then this map is injective, with image the subset of those functions satisfying the property stated in Theorem 4.0.2(2).

EXAMPLE 4.0.4 (The 2-torus as a quotient space). We now describe an alternative viewpoint on the 2-torus; but first let us consider the circle as a warmup. Consider the interval $[0, 1]$. There is an equivalence relation on $[0, 1]$ where the only identification we have make is $0 \sim 1$ (write this out precisely!). The resulting space is then homeomorphic to $\mathrm{S}^1$.

Now, consider $\mathrm{I} \times \mathrm{I} \subset \mathbb{R}^2$. Define an equivalence relation where we set points of the form $(x, 1) \sim (x, 0)$ for $x \in [0, 1]$ and $(0, y) \sim (1, y)$ for $y \in [0, 1]$ and $(a, b) \sim (a, b')$ if and only if $(a, b) = (a', b')$ for all other points. Drawing this out, we get a space which is homeomorphic to the 2-torus.

REMARK 4.0.5 (Categorical properties of quotients). We remark that the quotient topology on $\mathrm{X}/\sim$ is the finest topology on this set that makes $q : \mathrm{X} \to \mathrm{X}/\sim$ continuous.

It will be useful to speak of maps which are "of the form $\mathrm{X} \to \mathrm{X}/\sim$."

DEFINITION 4.0.6. A **quotient map** $f : \mathrm{X} \to \mathrm{Y}$ is a continuous surjection such that $\mathrm{U} \subset \mathrm{Y}$ is open if and only if $f^{-1}(\mathrm{U}) \subset \mathrm{X}$ is also open.

We note that if $f$ is a surjective map of sets, then we can define the following equivalence relation on X. We declare that X is partitioned into $\mathrm{X}_y := f^{-1}(y)$ for all $y$ (recall that an equivalence relation on a set is the same thing as partitioning the set); we set $\sim$ to be this equivalence relation. The universal property of a quotient space then furnishes a continuous bijection $\mathrm{X}/\sim \to \mathrm{Y}$ (it is evidently a bijection and the topology on $\mathrm{X}/\sim$ is such that it is continuous). From the description of the topology on Y, we see that Y is finer than $\mathrm{X}/\sim$ but then this means, by Remark 4.0.5, that they are homeomorphic.

**4.1. Group actions.** There is a wealth of topological spaces that are themselves groups, or related to groups. Recall that a group action on a set X is a map

$$m : \mathrm{G} \times \mathrm{X} \to \mathrm{X}$$

such that $m(e, x) = x$ when $e \in \mathrm{G}$ is the neutral element and $m(g', m(g, x)) = m(gg', x)$. The **orbit set** of this action is then quotient of X by the equivalence relation: $x \sim x'$ if and only if $x' = gx$ for some $g \in \mathrm{G}$. We write X/G as the quotient set which is simply X modulo this equivalence relation. We h ave the quotient map:

$$q : \mathrm{X} \to \mathrm{X}/\mathrm{G}.$$

Now a **topological group** is a group in the world of spaces: it is a group such that the multiplication $m : \mathrm{G} \times \mathrm{G} \to \mathrm{G}$ and the inverse: $\iota : \mathrm{G} \to \mathrm{G}, g \mapsto g^{-1}$ are continuous functions. A group action is then **continuous** if $m$ is a continuous map. In such a situation, then $q : \mathrm{X} \to \mathrm{X}/\mathrm{G}$ fits in the formalism of quotient spaces above.

Example 4.1.1 (Torus as a quotient). We have an action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ given by

$$\mathbb{Z}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \qquad (m,n) \times (x,y) \mapsto (x+m, y+n).$$

Then the quotient $\mathbb{R}^2/\mathbb{Z}^2$ is homeomorphic to $\mathbb{T}^2$.

Example 4.1.2 (Projective spaces). The real projective space $\mathbb{RP}^n$ is the space of "lines in $\mathbb{R}^{n+1}$ through the origin." To make this precise, we think of $\mathbb{R} \smallsetminus \{0\}$ as a group under multiplication. We have the following action of $\mathbb{R} \smallsetminus \{0\}$ on $\mathbb{R}^{n+1} \smallsetminus \{0\}$

$$\mathbb{R} \smallsetminus \{0\} \times \mathbb{R}^{n+1} \smallsetminus \{0\} \to \mathbb{R}^{n+1} \smallsetminus \{0\} \qquad r \times (x_0, \cdots, x_n) \mapsto (rx_0, \cdots, rx_n).$$

The quotient is denoted by

$$\mathbb{RP}^n := \mathbb{R}^{n+1} \smallsetminus \{0\}/\mathbb{R} \smallsetminus \{0\}.$$

Points in $\mathbb{RP}^n$ (in other words, equivalence classes) are denoted by $[x_0 : x_1 : \cdots : x_n]$; this refers to the equivalence class of the point $(x_0, \cdots, x_n)$ in $\mathbb{R}^{n+1} \smallsetminus \{0\}$.

There is yet another, arguably simpler way to think of $\mathbb{RP}^n$ as a quotient. Consider the antipodal action of the group of order 2, $\mathrm{C}_2$ on $\mathrm{S}^n$; this is described by

$$\mathrm{S}^n \to \mathrm{S}^n \qquad x \mapsto -x.$$

Then we can take the quotient $\mathrm{S}^n/\mathrm{C}_2$. In the homework you will be asked to prove rigorously that this is homeomorphic to $\mathbb{RP}^n$.

Example 4.1.3 (Configuration spaces and symmetric powers). Let X be a space, then the space $\mathrm{X}^{\times n}$ admits a continuous action of the symmetric group on $n$-letters $\Sigma_n$. The quotient space

$$\mathrm{X}^{\times n}/\Sigma_n := \mathrm{Sym}^n(\mathrm{X})$$

is called the **$n$-th symmetric power of** X. Recall that if G acts on a set X and $x \in \mathrm{X}$ its **stabilizer**[1] is the subgroup

$$\mathrm{Stab}_x := \{g \in \mathrm{G} : gx = x\}.$$

Points in X usually have nontrivia stabilizer: for example elements in the diagonal have $\mathrm{Stab}_{(x,\cdots,x)} = \Sigma_n$ since permutation does not move these elements. We will examine situations where we will like the action to be **free**: that is to say that $\mathrm{Stab}_x = \{e\}$ for all $x \in \mathrm{X}$. For example, if we consider the **configuration space on $n$-letters**

$$\mathrm{PConf}_n(\mathrm{X}) := \{(x_1, \cdots, x_n) : x_i \neq x_j \text{ if } i \neq j\}$$

then the action of $\Sigma_n$ is free and the quotient $\mathrm{Conf}_n(\mathrm{X}) = \mathrm{PConf}_n(\mathrm{X})/\Sigma_n$ is called the **unordered configuration space on $n$-letters**.

**4.2. CW complexes.** A CW complex is a way to "present" nice topological spaces in terms of simpler ones. Surely, whatever nice means, the spheres must be nice. The spheres are the boundary of the disk: those vectors which are $\leqslant 1$. We write this as

$$\mathrm{S}^n = \partial \mathrm{D}^{n+1}.$$

In this situation, we also consider $n = 0$; the boundary of $\mathrm{D}^1 = [0,1]$ are just two points 0 and 1. The language of quotients allows us to "attach disks along maps" as made formal by the following concept:

Construction 4.2.1. Let X be a space. Consider a continuous function $\varphi : \mathrm{S}^{n-1} \to \mathrm{X}$. By a **cell attachment in dimension** $n$, we mean the following: take the disjoint union

$$\mathrm{D}^n \sqcup \mathrm{X}$$

[1]You might remember from elementary group theory that stabilizers are complementary to orbits: say X is a finite set and G is a finite group then $\mathrm{Orb}_x := \{gx : g \in \mathrm{G}\}$ and that the quantity

$$|\mathrm{Orb}_x||\mathrm{Stab}_x| = |\mathrm{G}|$$

must be conserved.

and consider the equivalence relation generated[2] by $x \sim \varphi(x)$ whenever $x \in \mathrm{S}^{n-1}$. The resulting quotient space is denoted by

$$\mathrm{D}^n \sqcup \mathrm{X}/\sim =: \mathrm{X} \cup_\varphi \mathrm{D}^n.$$

Construction 4.2.1 is now used to produce spaces systematically.

Construction 4.2.2 (CW complexes). Produce a space X in the following way. First let $\mathrm{X}^0$ be a set of points, with the discrete topology. Say $\mathrm{X}^{n-1}$ has been constructed. We are given a collection of maps $\{\varphi_\alpha : \mathrm{S}^{n-1} \to \mathrm{X}^{n-1}\}_{\alpha\in \mathrm{A}}$. We can perform Construction 4.2.1 simultaneously: take the disjoint union $\mathrm{X}^{n-1} \sqcup \coprod_\alpha \mathrm{S}^{n-1}$ and impose the quotient as in Construction 4.2.1. We then construct an inductive sequence of spaces

$$\mathrm{X}^0 \subset \mathrm{X}^1 \subset \cdots \mathrm{X}^n \subset \cdots.$$

We define

$$\mathrm{X} := \bigcup \mathrm{X}^n$$

and topologize it as follows: a subset $\mathrm{U} \subset \mathrm{X}$ is open if and only if $\mathrm{U} \cap \mathrm{X}^n$ is open for all $n$. A **(strict) CW complex** is a space X homeomorphic to one constructed by this procedure.

For each $n$ and for each $\alpha$ we have the **characteristic map**

$$\Phi_\alpha : \mathrm{D}^n \hookrightarrow \mathrm{X} \sqcup \mathrm{D}^n \to \mathrm{X}^n \to \mathrm{X}$$

where the first map is the inclusion of the $\mathrm{D}^n$ indexed by $\alpha$. The data of $(\mathrm{X}^n, \Phi_\alpha)$ is called a **CW decomposition** of X.

Remark 4.2.3. In the literature, a CW complex is one that is *homotopy equivalent* to a strict CW complex. In fact the idea of CW complex is most powerfully allied with the notion of homotopy, which we will introduce later in class.

For now, we contend ourselves with describing some examples of CW complexes. Usually it is easier to draw them!

Example 4.2.4 ($\mathrm{S}^1$ as a CW complex). Let $\mathrm{X}^0 = \{x\}$ be a singleton . We have a map $\mathrm{S}^0 \to \{x\}$ given by the constant map (the only map!). Given this data, we construct $\mathrm{S}^1$ as a CW complex by cell attachment in dimension 1.

Example 4.2.5 ($\mathrm{S}^n$ as a CW complex). More generally, start with $\mathrm{X}^0 = \{x\}$ a singleton. We have the constant map $\mathrm{S}^{n-1} \to \{x\}$. Given this data, we can construct $\mathrm{S}^n$ as a CW complex by cell attachment in dimension $n$. Alternatively, there is also a CW structure on the sphere with 2-cells in each dimension. For example: we start with $\mathrm{S}^1$ and then consider two maps $\mathrm{S}^1_\mathrm{N}, \mathrm{S}^1_\mathrm{S} \to \mathrm{S}^1$ which are both the identity. Performing cell attachment for each cell in dimension 2 attaches the "north" and "south" hemisphere to create $\mathrm{S}^2$.

To be more precise: we start with $\mathrm{X}^0 = \{+, -\}$ with 2-points. In the first stage, take two copies of the identity map $\mathrm{S}^0_\mathrm{N}, \mathrm{S}^0_\mathrm{S} \to \mathrm{X}^0$ and perform cell attachment in dimension 1. We then get $\mathrm{S}^1$. Suppose that $\mathrm{X}^{n-1}$ has been built as a copy of $\mathrm{S}^{n-1}$. Again take two copies of $\mathrm{S}^{n-1}_\mathrm{N}, \mathrm{S}^{n-1}_\mathrm{N} \to \mathrm{X}^{n-1}$ and perform cell attachment in dimension $n$ to get $\mathrm{S}^n$.

Example 4.2.6 ($\mathbb{RP}^n$ as a CW complex). There is an inductive way of defining $\mathbb{RP}^n$ which produces a CW decomposition of it. Recall that $\mathbb{RP}^n$ is $\mathrm{S}^n/\mathrm{C}_2$ given by the antipodal action. We thus have the quotient map $\mathrm{S}^n \to \mathbb{RP}^n$ on which we can perform cell attachment in dimension $n+1$ and the resulting space is homeomorphic to $\mathbb{RP}^{n+1}$. This will be in the homework.

[2]in this situation, we just declare that points in X which are not in the image of $\varphi$ or points in $\mathrm{D}^n \setminus \mathrm{S}^{n-1}$ are in their own equivalence class.

CHAPTER 3

# Connectedness, Hausdorfness and compactness

In analysis, we have seen that ideas of connectedness and compactness are ubiquitous. For example, we have the *intermediate value theorem*: if $f : [a, b] \to \mathbb{R}$ is a continuous function and $r$ is a real number between $f(a)$ and $f(b)$, then there exists a $c \in [a, b]$ that achieves $r$, i.e., $f(c) = r$. We also have the *maximum value theorem*: if $f : [a, b] \to \mathbb{R}$ is continuous then there exists $c \in [a, b]$ such that $f(x) \leqslant f(c)$ for all $x \in [a, b]$. It turns out that there are abstract versions of these results for an arbitrary topological space and they illustrate the concepts of connectedness and compactness which are ubiquitous in the subject and, as it turns out, in a lot of mathematics. Another concept that we have decided to tie together with compactness is the idea of Hausdorffness. This corresponds to the basic spatial idea that points can be separated using open sets in the world of analysis.

## 1. Connectedness

Definition 1.0.1. A **separation** of a space $\mathrm{X}$ is a pair $\mathrm{U}, \mathrm{V}$ of disjoint open, nonempty subsets such that $\mathrm{U} \sqcup \mathrm{V} = \mathrm{X}$. A space $\mathrm{X}$ is said to be **connected** if there does not exist a separation of $\mathrm{X}$.

In practice, finding a separation can be quite hard as you will need both U and V to cover X. In practice, proving that a space is connected can be accomplished in two ways:

(1) show that is **path connected** — this will work for many “nice” topological spaces of interest;
(2) producing non-trivial clopen subsets.

Let us introduce this latter concept: a subset of X is said to be **clopen** if it is both closed and open.

Lemma 1.0.2. *A space* $\mathrm{X}$ *is connected if and only if the only clopen in* $\mathrm{X}$ *are only* $\emptyset$ *and* $\mathrm{X}$.

Proof. Note that a separation of X is clopen: U is assumed to be open but it is the complement of V and hence also closed. Therefore if a separation exists, there are clopen’s which are not $\emptyset$ or X. On the other hand, if A is clopen, then $\mathrm{X} \smallsetminus \mathrm{A}$ and A form a separation. □

We thus say that a clopen subset is nontrivial if it is neither $\emptyset$ nor all of X. The first result we want to prove is the following generalized intermediate value theorem

Theorem 1.0.3 (Generalized intermediate value theorem). *Let* $f : \mathrm{X} \to \mathrm{Y}$ *be a continuous function where* $\mathrm{X}$ *is connected and* $\mathrm{Y}$ *has the order topology. Then, if* $a, b$ *are two points of* $\mathrm{X}$ *and* $r \in [f(a), f(b)]$ *then there exists a* $c \in \mathrm{X}$ *such that* $f(c) = r$.

We first begin with the following simple observation.

Lemma 1.0.4. *The image of a connected space under a continuous function is connected.*

Proof. Since we are trying to prove something about the image of a continuous function, we might as well assume that $f : \mathrm{X} \to \mathrm{Y}$ is surjective, X is connected and prove that Y is connected. Now, let U, V be a separation of Y. Then $f^{-1}(\mathrm{U})$ and $f^{-1}(\mathrm{V})$ are separations of X (key point: since $f$ is surjective they are nonempty), contradicting the fact that X was assumed to be connected.

□

THEOREM 1.0.5 (Generalized intermediate value theorem). *Let $f : X \to Y$ be a continuous function where $X$ is a connected topological space and $Y$ be a totally ordered set with the order topology. Let $a, b \in X$ such that $f(a) < f(b)$ and let $r \in Y$ such that $r \in (f(a), f(b))$, then there exists an $c \in X$ such that $f(c) = r$.*

PROOF. Consider the set

$$A := f(X) \cap (-\infty, r) \qquad B = f(X) \cap (r, +\infty).$$

Then A and B are evidently disjoint and nonempty (one contains $f(a)$ the other contains $f(b)$). Furthermore, as subspaces of $f(X)$ (with the subspace topology) they are both open.

Suppose that there is no $c$ for which $f(c) = r$. Then we conclude that $f(X) = A \sqcup B$ and thus is a separation of $f(X)$. But this contradicts Lemma 1.0.4 □

In order to make use of the IVT and recover the analysis version of the theorem we need the following result.

THEOREM 1.0.6. *Let $X$ be a totally ordered set with the least upper bound property and assume that for $x < y$ there exists a $z$ such that $x < z < y$ (such a set is called a **linear continuum**). Then, with the order topology, $X$ is connected and so are intervals and rays*[1]*. In particular, intervals and rays in $\mathbb{R}$ are connected.*

PROOF. We take this for granted for now. But we will give an alternate proof once we have learned something about compactness.

□

**1.1. Path connectedness.** Having Theorem 1.0.6 we can formulate another notion of connectedness which is often the correct one in practice.

DEFINITION 1.1.1. Let X be a space and $x, y \in X$. Then a **path (between $x$ and $y$)** is a continuous function $f : [a, b] \to X$ such that $f(a) = x, f(b) = y$. A space is path connected if every point of X can be jointed by paths in X.

LEMMA 1.1.2. *Any path connected space is connected.*

PROOF. If $X = A \sqcup B$ is a separation, then by Lemma 1.0.4, any path $f : [a, b] \to X$ must have image entirely be contained in A or B because, otherwise, $A \cap f([a, b])$ and $B \cap f([a, b])$ is a separation of the image. But this means that points in A cannot be connected to points in B. □

There are spaces which are connected but not path connected. We will give the usual example — the topologist's sine curve. But we will only do that once we have discussed closures.

## 2. Hausdorffness

The next topic of interest involves what is called in the literature as "separation axioms." There is a certain notoriety to this whole subject — it involves a bunch of terminology which is hard to commit to memory and mostly concerns pathological topological spaces that do not appear in nature. We will not really indulge on phylogeny of spaces in this course. Rather we will work towards motivating and studying the most important separation condition: Hausdorffness (see Definition 2.2.1).

[1]A ray means a set of the form $(-\infty, a)$ or $(a, +\infty)$

**2.1. Closures.** To see the importance of Definition 2.2.1 let us discuss closed sets in more detail. So far, we have emphasized more on the concept of an open set, but we have also noted that closed sets could have been used to define topologies. Analysis tells us that closed sets are those which contains their own limit points: in other words we can take a set and adjoin all of its limits points to get its closure. The abstract definition of closure is given as follows.

Let $\mathrm{A} \subset \mathrm{X}$ be a subset; then its **closure** is defined to be the

$$\overline{\mathrm{A}} = \bigcap_{\mathrm{A} \subset \mathrm{Z},\ \mathrm{Z} \text{ is closed in } \mathrm{X}} \mathrm{Z}.$$

Evidently, $\mathrm{A} \subset \mathrm{X}$ is closed if and only if $\mathrm{A} = \overline{\mathrm{A}}$.

Remark 2.1.1 (Boundaries and interiors). Other than closures, there are at least two other subspaces of X that one might be interested in. We have the **interior** of A

$$\mathrm{A}^{\circ} := \bigcup_{\mathrm{U} \subset,\ \mathrm{U} \text{ is open in } \mathrm{X}} \mathrm{U};$$

as well as its boundary

$$\partial \mathrm{A} := \overline{\mathrm{A}} \cap \overline{\mathrm{X} \smallsetminus \mathrm{A}}.$$

Clearly, $\mathrm{A}^{\circ} \subset \mathrm{A} \subset \overline{\mathrm{A}}$ and one can show $\overline{\mathrm{A}} = \mathrm{A}^{\circ} \cup \partial \mathrm{A}$. These are somewhat less important than closures of A.

In general, an important question in topology is the following:

Question 2.1.2. Given a set $\mathrm{A} \subset \mathrm{X}$ that we are interested in, how might we describe its closure?

Question 2.1.2 is often interesting. The first instance of trying to compute closures was encountered in real analysis:

Example 2.1.3 (Rational approximation). Consider $\mathbb{Q} \subset \mathbb{R}$. Then a standard result in real analysis says the following: for any $r \in \mathbb{R}$ and any $\epsilon > 0$, there exists $q \in \mathbb{Q}$ such that $|q - r| < \epsilon$. This is the basic fact that any real number can be written in terms of a decimal expansion up to an arbitrarily large degree of accuracy. The closure of $\mathbb{Q}$ in $\mathbb{R}$ is then $\mathbb{R}$ itself; an easy way to see this is given by Proposition 2.1.6. This suggests that one way to think of closures is to include all points which approximate points in A up to an arbitrary large degree of accuracy. We make sense of this latter notion next.

Example 2.1.4 (Sierpinski space). Let $\mathrm{D} = \{\eta, x\}$ (the names are evocative of their standard names in algebraic geometry). The **Sierpinski space** is the topology on D where the only opens are declared to be $\emptyset, \mathrm{D}$ and $\eta$. The reader is encourages to check that this is indeed a topology. We observe that $x$, being the complement of D, is a closed point while $\eta$ is is an open point. What is somewhat strange is that $\eta$ is *not* a closed point: its closure is evidently all of D. So one writes something unfamiliar: $\overline{\{\eta\}} = \mathrm{D}$. This space (and other variants) is actually quite important in algebraic geometry but more as a "bookkeeping device."

Taking a cue from real analysis one might think to describe closures via the notion of a **limit point**: if $\mathrm{A} \subset \mathrm{X}$ is a subset and $x \in \mathrm{X}$ then we say that $x$ is a limit point of A if

$$x \in \overline{\mathrm{A} \smallsetminus \{x\}}.$$

This captures the idea that points of A clusters around $x$. The following lemma is a helpful way to visualize this condition

Lemma 2.1.5. *Let* $\mathrm{A} \subset \mathrm{X}$ *be a subset, then* $x \in \overline{\mathrm{A}}$ *if and only if every open neighborhood of* $x$ *intersects* A *nontrivially.*

PROOF. We claim that $x \notin \overline{A}$ if and only if there exists an open in X that contains $x$ but does not intersect A. If $x \notin \overline{A}$, then the set $U := X \setminus \overline{A}$ is open, being the compement of a closed, and does not intersect A. On the other hand if U is an open that contains $x$ and does not intersect A, then the closed set $X \setminus U$ must contain A. Since the closure is given as all the intersection of closed that contains A, we have that $\overline{A} \subset X \setminus U$. But if $x \in \overline{A}$ then it must be in $X \setminus U$ contradicting the fact that $x$ is contained in U. □

Therefore, a point $x \in X$ is a limit point of X if and only if every open neighborhood of $x$ intersects A at a point other than itself. The following result "computes" the closure of any set A.

PROPOSITION 2.1.6. *Let* $A \subset X$ *be a subset and* $A'$ *the set of all limit points of* X. *Then* $\overline{A} = A \cup A'$. *Therefore a set is closed if and only if it contains all of its limit points. Therefore, a subset* A *is closed if and only if it contains all of its limit points.*

PROOF. We first claim that the closure must contain the limit points. If $x \in A'$ then, by definition, every neighborhood of $x$ intersects A in a point different from $x$ itself. Applying Lemma 2.1.5 we learn that $x$ must be in $\overline{A}$. Hence $A \cup A' \subset \overline{A}$.

Now, take $x \in \overline{A}$. If $x \in A$ itself we are done. If it does not lie in A but lie in $\overline{A}$ then Lemma 2.1.5 again says that every open of $x$ intersects A trivially; but since $x \notin A$ we have that it intersects A in a point different from $x$. Therefore, $x$ is a limit point of A and we are again done. □

Therefore the task of describing closures boils down to understanding limit points of A; this sounds just like analysis. The following definition will be useful in describing limit points and abstracts one of the first ideas encountered in real analysis.

DEFINITION 2.1.7. A sequence of points $\{x_n\}_{n \in \mathbb{N}}$ is said to **converge** to $x \in X$ if for any open neighborhood U of $x$ there exists an $N \gg 0$ such that for all $n \geqslant N$, $x_n \in U$.

However, we see that there are two phenomenon which makes limit points rather unwieldy for general topological spaces:

(1) points may not be closed; we have seen this in the Sierpinski space from Example 2.1.4. This is very much counter to our experience from real analysis where points in $\mathbb{R}^n$ with the Euclidean topology are closed.
(2) Limit points may not be unique. An infamous example is the **bug-eyed line**. Take $\mathbb{R}_+ \sqcup \mathbb{R}_-$ and define the equivalence relation given by $r_+ \sim r_1$ for $r_+ \in \mathbb{R}_+ \setminus \{0\}$ and $r_- \in \mathbb{R}_- \setminus \{0\}$. One should imagine that we have glued two copies of the real line except at the point 0 where it has been doubled; call this space B. One can check that $\mathbb{R} \setminus 0 \subset B$ has both $0_+$ and $0_-$ as limit points.

The idea of Hausdorfness removes these two phenomenon. However we note, contrary to what is said in textbooks, these spaces are still very interesting.

EXAMPLE 2.1.8 (Topologist's sine curve). We now explain a space which is connected but not path connected. Consider

$$S = \{(x, \sin(1/x)) : x < 0 \leqslant 1\}.$$

Then S is the image of $(0, 1] \to \mathbb{R}^2$ given by $x \mapsto \sin(\frac{1}{x})$ and hence is connected since it is the image of a connected set. By Lemma 2.1.9 the closure of a connected subspace is connected hence the space $\overline{S}$ is connected; in fact as a set it is equal to S and the interval $0 \times [-1, 1]$.

We claim that it is not path connected. Let $\gamma : [a, b] \to \overline{S}$ be a path starting at the origin. This path can choose to get "stuck" at $0 \times [-1, 1]$ for a while or enter S. The collection of points $x \in [a, b]$ which is stuck at $0 \times [-1, 1]$ is the inverse image of a closed subset $0 \times [-1, 1] \subset \overline{S}$ and hence is closed. Therefore we may reparametrize the path such that we have $\gamma : [a, b] \to \overline{S}$ for which $\gamma(a) \in 0 \times [-1, 1]$ but $\gamma((a, b]) \in S$. We again reparametrize to get $\gamma : [0, 1] \to \overline{S}$

In this case we have that $\gamma(t) = (x(t), y(t))$ where $x(t) = 0$ for $t = 0$, and $x(t) > 0$, $y(t) = \sin(1/x(t))$ for $t > 0$; each $x(-)$ and $y(-)$ are continuous functions since $\gamma$ is. Now, let us claim that there exist a sequence $t_n \mapsto 0$ such that $y(t_n) = (-1)^n$. This produce a convergent sequence that gets mapped into a non-convergent sequence, contradicting continuity (since the limit point 0 is in the domain). For each $n$, find a $u$ such that $0 < u < x(1/n)$ for which $\sin(1/u) = (-1)^n$. Using the intermediate value theorem we then find a $t_n$ with $0 < t_n < 1/n$ such $x(t_n) = u$.

LEMMA 2.1.9. *The closure of a connected space is connected. More generally if* X *is a space,* A *is a connected subspace of* X*, and we have* $A \subset B \subset \overline{A}$ *then* B *is connected.*

PROOF. First, we claim the following: if $X \subset Y$ is a subspace inclusion and X is connected and Y is separated by U and V then either X is in U or in V. Indeed, otherwise, $X \cap U$ and $X \cap V$ would constitute a separation of X.

Now, let U and V be a separation of B. Then, since A is connected, we must have that $A \subset U$ or $A \subset V$. We assume that $A \subset U$. Now V does not meet A but covers $\overline{A}$. Hence it must meet contain some limit point of A by Proposition 2.1.6. But, by the definition of a limit point and the fact that V is open in B we must have that $V \cap A \neq \emptyset$. This is a contradiction. □

**2.2. Hausdorff topological spaces.** The right axiom to ensure that these two phenomenon does not arise was isolated by Felix Hausdorff.

DEFINITION 2.2.1. A space X is said to be **Hausdorff** or **separated** if for any two distinct points $x, y$ there exists open neighborhoods $x \in U_x$, $y \in U_y$ such that $U_x \cap U_y = \emptyset$.

LEMMA 2.2.2. *Let* X *be a Hausdorff space. Then:*

*(1) every finite set of points in* X *is a closed set;*
*(2) limit points of sequences are unique.*

PROOF. It suffices to check that singletons are closed, which means that we must prove that $U := X \setminus \{x\}$ is open. But now for each $x_0$ different from $x$ we can find an open V such that $x_0 \in V$ and V does not contain $x$ (since we can also "engulf" $x$ in its own open not intersecting V). Therefore $x_0 \in V \subset U$. Since we can do this for all points of U, it must be open and hence $\{x\}$ is closed.

We prove the second point. Let $x_n$ be sequence converging to $x$. Suppose it converges to $y \neq x$. Take $U_y$ an open containing $y$ and $U_x$ an open containing $x$ disjoint from it. By the definition of limit points, all but finitely many $x_n$'s are contained inside $U_x$. Hence only finitely many $x_n$'s can be in $U_y$ and thus cannot converge to $y$. □

EXAMPLE 2.2.3. The subspace of a Hausdorff space is Hausdorff and arbitrary products Hausdorff spaces is Hausdorff. Furthermore, totally ordered sets (with the order topology) are Hausdorff. These are easy consequences of the axioms and tells us that a lot of the spaces we have encountered so far are Hausdorff.

EXAMPLE 2.2.4. The triangle inequality guarantees that metric spaces are Hausdorff. Hence Euclidean spaces and subspaces thereof are all Hausdorff.

EXAMPLE 2.2.5. Quotients are typically not Hausdorff! As exemplified by the bug-eyed line. Proposition 2.2.6 gives a way to think about those quotients which remain Hausdorff.

The next proposition is a nice characterization of Hausdorff spaces and, in some context, a better definition.

PROPOSITION 2.2.6. *Let* X *be a space, then it is Hausdorff if and only if the diagonal* $\Delta(X) \subset X \times X$ *is a closed subspace.*

Proof. Let $x \neq y$. Consider $(x,y) \in X \times X$. Since the latter is not in the diagonal there exists an open of the form $U \times V \subset X \times X$ such that $U \times V \cap \Delta(X) = \emptyset$ and $U, V \subset X$ are open. But this means that $\Delta^{-1}(U \times V) = U \cap V = \emptyset$.

On the other hand, suppose that X is Hausdorff. Consider $(x,y) \in X \times X \setminus \Delta(X)$. Then $U_x \times U_y$ wher $U_x, U_y$ are opens containing $x$ and $y$ respectively such that they do not intersect furnishes an open neighborhood of $(x,y)$ contained in $X \times X \setminus \Delta(X)$. □

We can use this to give a criterion for when a quotient of a Hausdorff remain Hausdorff. Recall that an equivalence relation on X is just a certain subset $R \subset X \times X$.

Proposition 2.2.7. *Let* X *be a Hausdorff, then* $X/\sim$ *is Hausdorff if and only if* $R \subset X \times X$ *is closed.*

Proof. By Proposition 2.2.6, we have that $X/\sim$ is Hausdorff if and only $\Delta(X/\sim)$ is closed. Consider the map $q \times q : X \times X \to X \times / \sim \times X/\sim$; this map satisfies: a set U in the target is open if and only if its preimage is open (since this is the case for the quotient map $q$). Notice that $(q \times q)^{-1}(\Delta(X/\sim)) = R$. Hence $\Delta(X/\sim)$ is closed if and only if R is closed.

□

Here's another useful proposition that we can prove using the diagonal perspective. We say that a subset $U \subset X$ is **dense** if $\overline{U} = X$.

Proposition 2.2.8 (Density lemma). *Let* $g, f : X \to Y$ *be two continuous function such that* $f|_U = g|_U$ *where* $U \subset X$ *is dense. If* Y *is Hausdorff, then* $f = g$.

Proof. Consider $\Gamma_{f,g} := \{x : f(x) = g(x)\} \subset X$. Then $U \subset \Gamma_{f,g}$. Since we know that $\overline{U} = X$, it suffices to prove that $\Gamma_{f,g} \subset X$ is closed to conclude that $\Gamma_{f,g} = X$ which is what we want to show. Consider the map $(f,g) : X \to Y \times Y$. Then $(f,g)^{-1}(\Delta(Y)) = \Gamma_{f,g}$ by definition. Hence $\Gamma_{f,g}$ is closed as soon as we know that $\Delta(Y)$ is closed. This is equivalent to being Hausdorff by Proposition 2.2.6. □

**2.3. More separation axioms and Urysohn's lemma.** While the Hausdorff condition is often satisfied in practice, we will examine a couple of other separation axioms which will leads to some very useful constructions in topology. The most important of which is the idea of a **normal space**. It arises from the idea that we want to create continuous "indicator functions" on a set. Evidently, indicator functions are far from continuous unless the domain is something like a space with the discrete topology. We introduce a couple of new ideas.

The **support** of a continuous function $f : X \to \mathbb{R}$ is the closure of the set of points where $f$ is nonzero:

$$\mathrm{supp}(f) := \overline{\{x : f(x) \neq 0\}}.$$

Hence, if $x$ is not in the support, there exists a neighborhood of it, say U, such that $f|_U = 0$. If A is a closed subset of X, then a **bump function of** A is a function whose support is A[2]. For example you might have seen the Gaussian, $G : \mathbb{R} \to \mathbb{R}$:

$$G(x) = \begin{cases} \exp(-\frac{1}{1-x^2}) & |x| < 1 \\ 0 & |x| \geqslant 1. \end{cases}$$

In this case, the support of G is exactly the interval $[-1, 1]$. The utility of a bump function is that if we have a continuous function $f : U \to \mathbb{R}$, then we can extend it to all of of X by taking a bump function $\varphi : X \to \mathbb{R}$ whose support A is contained inside U and then define

$$(f \cdot \varphi)(p) = \begin{cases} f(p) \cdot \varphi(p) & \text{if } p \in U \\ 0 & \text{else.} \end{cases}$$

[2]In the literature bump functions usually are requested to have compact support and its support to exactly be a compact set A.

The new function $f \cdot \varphi$ enjoys the property that it is a continuous function defined on all of X and does not alter the values of the original function $f$ inside A.

For the rest of this discussion, we will consider continuous functions $f : \mathrm{X} \to [0,1]$ in which case by the support we mean

$$\operatorname{supp}(f) := \overline{f^{-1}((0,1])}.$$

This leads to the following definition.

Definition 2.3.1. Let X be a topological space and $\{\mathrm{U}_i\} = \mathcal{U}$ is a cover, assumed to be *finite*. Then a **(finite) partition of unity subordinate to** $\mathcal{U}$ is a collection of continuous functions

$$\psi_i : \mathrm{X} \to [0,1]$$

(1) $\operatorname{supp}(\psi_i) \subset \mathrm{U}_i$;
(2) $\sum_{i=1}^{n} \psi_i(x) = 1$ for all $x \in \mathrm{X}$.

Partitions of unity are very useful in practice: in manifold theory it helps us embed an abstract manifold into a big Euclidean space, in algebraic topology they reduce problems about topological gadgets like vector bundles into the "universal case." For now, we note that if we are given functions $f_i : \mathrm{U}_i \to \mathbb{R}, i = 1 \cdots n$, then we can construct a new continuous function $\Phi := (\varphi_1, \cdots, \varphi_n) : \mathrm{X} \to \mathbb{R}^n$ given by

$$\varphi_i := f_i \cdot \psi_i.$$

If each $f_i$ was assumed to be injective and homeomorphic to its own image then $\Phi$ is itself satisfies this condition! This means that we can regard X as a subspace of Euclidean space from which we can deduce a whole whose of properties about X — for example it must be metrizable.

Hence we ask the following question:

Question 2.3.2. When does a partition of unity exist?

The right generality is the following, a strengthening of the Hausdroff axiom.

Definition 2.3.3. A **normal space** is a space X such that:

(1) points are closed;
(2) for any two closed subsets A and B, there exists disjoint opens containing A and B respectively.

To motivate the definition of a normal space, we remark that when we want to naively construct an indicator function, we care about the inverse of image of 1 (the support, i.e., the set that we want to "indicate") and the inverse image of 0 (the rest). Because indicator functions are typically not continuous, we should be mapping into the interval $[0,1]$ instead of the discrete set $\{0,1\}$ and we are primarily interested in the inverse images $f^{-1}(1)$ and $f^{-1}(0)$ which are closed subsets of X; note that $f^{-1}(0)$ cannot typically be the complement of $f^{-1}(1)$ if we want $f$ to be continuous. But since we have a continuous function $f : \mathrm{X} \to [0,1]$ the subsets $f^{-1}(0)$ and $f^{-1}(1)$ are separated by opens $f^{-1}([0,1/2))$ and $f^{-1}((1/2,1])$ respectively. Hence it is natural to ask for the disjoint open condition in a normal space.

Lemma 2.3.4. *A normal space is Hausdorff.*

Proof. Follows from the fact that points are assumed to be close. □

There is an intermediate hypothesis between normality and Hausdorffness called **regular**: this means that 1) points are closed and 2) for a point $x$ and a closed subset B not containing $x$ there exists disjoint opens containing B and $x$ respectively. The next lemma reformulates the conditions of being normal and being regular in useful ways.

LEMMA 2.3.5. *Assume that* X *is a space such that points are closed. Then* X *is regular (resp. normal) if and only if for each* $x \in X$ *(each closed subset* $A \subset X$*) and each neighborhood* U *containing* $x$ *(each open set* U *containing* A*) there is an open set* V *containing* $x$ *such that* $\overline{V} \subset U$ *(an open set containing* A *such that* $\overline{V} \subset U$*).*

PROOF. The proof works in the same ways for both characterizations of normality and of regularity, so we say something about the proof in the case of regular spaces. The key point is that if $x$ is a point of X and U is a neighborhood of it, we take $B := X \smallsetminus U$ the closed subset complementary to U. Then we can find V containing $x$ and W containing B which are disjoint. This ensures that $\overline{V} \subset U$. The converse is easier. □

Every normal space is clearly a regular space but it need not be normal; we refer to [**Mun00**, 31, Example 2] for an example: take the product of $\mathbb{R}$ with the lower limit topology. Rather than focusing in non-examples, we now state the first important result in this course and give many examples of normal spaces in 2.5.

THEOREM 2.3.6 (Urysohn's Lemma). *Let* X *be any normal space if and only if for any* A *and* B *be disjoint closed subsets of* X*, there exist a continuous function*

$$f : X \to [a, b]$$

*such that* $f|_A = a$ *and* $f|_B = b$*.*

In a sense: Urysohn's lemma says that the disjoint opens separating A and B can be witnessed by an explicit function. We remark, however, that $f^{-1}(a)$ need not be A itself, and similarly for B.

PROOF. We will construct a sequence of *closed* sets $\{A_n\}_{0 \leqslant n = p/2k \leqslant 1}$ (so the indexing sets are numbers of the form an integer over $2^k$) with the following properties:

(1) $A_0 = A$
(2) $A_1 = X$
(3) $A_n \subset A_m^\circ$ for any $n < m$
(4) $A_n \subset X \smallsetminus B$ for $n < 1$.

We imagine that this is a sequence of sets, expanding from A to all of X (by (1) and (2)) such that (by (3)) the previous set is contained in the interior of the next set and (by (4)) it *does not* touch B until $n = 1$.

Taking this construction for granted, we will prove Urysohn's theorem. Set

$$f : X \to [0, 1] \qquad f(x) = \inf\{n : x \in A_n\}$$

This function takes $x$ and tells you the smallest index $n$ for which $x$ belongs to $A_n$. Evidently, $f|_A = 0$ and $f|_B = 1$ (by (4)). What is somewhat surprising is that this function is continuous.

Consider the interval $[0, a)$ for some $0 < a < 1$. Then

$$f^{-1}([0, a)) = \{x : f(x) < a\} = \bigcup_{n<a} A_n = \bigcup_{n<a} A_n^\circ,$$

because of (1) and (3). Only the last equality requires an explanation: suppose that $x \in A_n$ for some $n < a$. Then we just choose a number $p/2^k$ such that $n < m = p/2^k < a$ (which exists by density of such numbers!). Then by (3), we have that $A_n \subset A_m^\circ$ and hence we are done.

On the other hand, consider the interval $(b, 1]$ for $0 < b < 1$. Then

$$f^{-1}((b, 1]) = \bigcup_{n>b} X \smallsetminus A_n,$$

clearly a union of opens. Therefore, since these intervals generate the topology on $[0, 1]$ we are done.

The sequence of sets is best constructed via a picture (see [**Mun00**, Page 209]) and the key property of normality we use is the reformulation of Lemma 2.3.5. We first begin by setting $U_1 := X \smallsetminus B$ which is an open containing A. By Lemma 2.3.5 we find $U_0$ an open, containing

A, and such that $A_0 := \overline{U}_0 \subset U_1$. The next index that appears is 1/2. Set $U_{\frac{1}{2}}$ to be an open such that $A_0 \subset U_{\frac{1}{2}}$ and $\overline{U}_{\frac{1}{2}} \subset U_1$; this again uses Lemma 2.3.5 (applied to $A_0$) and $A_{\frac{1}{2}} := \overline{U}_{\frac{1}{2}}$. In the next step apply Lemma 2.3.5 to $A_{\frac{1}{2}}$. Repeating this construction constructs these $A_n$'s which satisfies (3) and (4). To ensure (1) and (2) we simply rename $A_0$ to be A and $A_1$ to be X (clearly harmless procedures). □

Sometimes we will use Urysohn's lemma in the form below (it is analogous to Lemma 2.3.5

COROLLARY 2.3.7. *Let* X *be a normal space,* $V, U \subset X$ *open subsets such that* $\overline{V} \subset U$*. Then there exists a function* $f : X \to [0,1]$ *such that* $f|_{\overline{V}} = 1$ *and* $f|_{X \setminus U} = 0$*.*

REMARK 2.3.8. Suppose that X was only a regular space. Note that Lemma 2.3.5 cannot guarantee that, given an inclusion $A \subset U$ where A is closed in X and U is open, another open V which still contains A and whose closure sits in between A and U. This is one of the key insights in the proof of Urysohn's lemma. Lemma 2.3.5 does let us do this when A is singleton — let us call this open $V_x$, however and maybe we can run Urysohn's lemma using $\bigcup_{x \in A} V_x$. Of course there is no way to guarantee that the closure of $\bigcup_{x \in A} V_x$ is still contained in U. We will see that this can be fixed given another hypothesis on X (second countability).

**2.4. Applications.** We now show how to use Urysohn's theorem to prove three important results in topology. The first result is an *embedding result*: it places an abstract topological space with nice properties in a familiar space. This is analogous to putting "coordinates" (aka basis) on an abstract vector space. We say that a map $f : X \to Y$ is an **embedding** if the $f$ is injective and the continuous bijection $f : X \to f(X)$ is a homeomorphism. In other words, X is "placed incide Y" as a subspace.

LEMMA 2.4.1. *A continuous injection* $f : X \to Y$ *is an embedding if and only if it is an open map.*

The goal of the next theorem embeds certain nice normal spaces into a metric space; more precisely into the Hilbert cube. Noting that subspaces of metrizable spaces are metrizable, we see that any space embeddable in a metric space must be metrizable.

THEOREM 2.4.2 (Urysohn's metrization theorem). *Let* X *be a normal space with a countable basis (taken literally: a basis for the topology consisting of countably many elements), then it is metrizable. More precisely: there is an embedding into the Hilbert cube:*

$$X \hookrightarrow [0,1]^{\mathbb{N}}.$$

We have seen, from the exercise in homework 2 that both interesting metrics (the Euclidean and uniform ones) on the Hilbert cube agree.

PROOF. Let $\mathcal{B} = \{B_i\}$ be a countable basis of X. Let B and B′ be a pair of basis elements; let us say that B **is engulfed in** B' if $\overline{B} \subset B'$. Then, Theorem 2.3.6 asserts that there exists a continuous function

$$f : X \to [0,1],$$

such that $f$ is 1 on B (because it is so on $\overline{B}$) and 0 outside of B′. Performing this procedure for each pair which are engulfed in each other produces countably many functions $f_1, f_2, \cdots$ (the indexing set changes from the indexing for $\mathcal{B}$ but it remains countable because it is at most of size $|\mathbb{N} \times \mathbb{N}|$). Set

$$\Phi : X \to [0,1]^{\mathbb{N}} \qquad x \mapsto (f_1(x), f_2(x), \cdots).$$

The function Φ is continuous and we need to prove that Φ is an emebdding. It is injective: if $x \neq y$ then, by normality, there exists a basis element B engulfed in another basis element B′ such that $x \in B$ and $y \notin B'$ and thus some component of Φ will tell the difference between these two elements (it is 1 on $x$ and zero on $y$). So we need only check, by Lemma 2.4.1 that Φ is an open map.

Let $\mathrm{U} \subset \mathrm{X}$ be an open and $\varphi(x) \in \varphi(\mathrm{U})$. We claim that there exists an open $\mathrm{V} \subset [0,1]^{\mathbb{N}}$ containing $\Phi(x)$ such that $\mathrm{V} \cap \Phi(\mathrm{X}) \subset \Phi(\mathrm{U})$; this ensures that $\Phi(\mathrm{U})$ is open. To do this, choose a basis elements $\mathrm{B}, \mathrm{B}'$ such that

$$x \in \mathrm{B} \subset \overline{\mathrm{B}} \subset \mathrm{B}' \subset \mathrm{U}, \tag{2.4.3}$$

and let $f_k$ be the function constructed as in the first paragraph for B and B′; so this means that $f_k(x) = 1$ and $f_k(\mathrm{X} \setminus \mathrm{B}) = 0$. Consider the open set $\mathrm{V} := (0,1] \times \prod_{i \neq k} [0,1]^{\mathbb{N}}$. This is the open set, in the product topology, given by $[0,1]$ in every coordinate except the $i$-th where it is the open set $(0,1]$. We claim that this V works. To do so we need to check that $\Phi(x) \in \mathrm{V}$ and that $\mathrm{V} \cap \Phi(\mathrm{X}) \subset \Phi(\mathrm{U})$.

Since $f_k(x) = 1$, we indeed have that $\Phi(x) \in \mathrm{V}$. Next, if $t \in \mathrm{V} \cap \Phi(\mathrm{X})$ then we have that $t = \Phi(y)$ for some $y \in \mathrm{X}$ such that $f_k(y) > 0$. This means that $y \notin \mathrm{X} \setminus \mathrm{B}'$ and hence is in B′ which means that it must be in U by (2.4.3).

□

Next, we mention Tietze's extension theorem. It helps us extend partially-defined functions: those whose domain is only a closed subspace of X. The proof will take us further afield but see [**Mun00**, 35]

THEOREM 2.4.4 (Tietze extension theorem). *Let* X *be a normal space and* $\mathrm{A} \subset \mathrm{X}$ *a closed subspace. Then any continuous function* $f : \mathrm{A} \to \mathbb{R}$ *extends to a continuous function* $\mathrm{X} \to \mathbb{R}$ *(in other words, there exists a continuous function* $\tilde{f} : \mathrm{X} \to \mathbb{R}$*) such that* $\tilde{f}|_{\mathrm{A}} = f$*).*

Nonetheless we note the following result:

PROPOSITION 2.4.5. *Assume the Tietze extension theorem. Then Urysohn's lemma holds.*

PROOF. Let X be as in the hypothesis of Urysohn's lemma. If A and B are the closed subsets in question then define

$$f : \mathrm{A} \cup \mathrm{B} \to [0,1]$$

by sending all of A to 0 and all of B to 1. Since A and B are disjoint, this is a continuous function. By Tietze's extension theorem, it extends to a continuous function on all of X! □

Lastly, we achieve one of the goals that we set out at the start of this discussion.

THEOREM 2.4.6 (Existence of a partition of unity). *Let* X *be a normal space equipped with a* $\mathcal{U}$*, a finite covering. Then there exists a partition of unity subordinate to* $\mathcal{U}$*.*

PROOF. We take for granted the refinement lemma below. Let $\mathcal{U} = \{\mathrm{U}_1, \cdots, \mathrm{U}_n\}$ be a partition of unity, then choose $\mathcal{V} = \{\mathrm{V}_1, \cdots, \mathrm{V}_n\}$ an open cover such that $\overline{\mathrm{V}}_i \subset \mathrm{U}_i$ and another open cover $\mathcal{W} = \{\mathrm{W}_1, \cdots, \mathrm{W}_n\}$ such that $\overline{\mathrm{W}}_i \subset \mathrm{V}_i$. By Corollary 2.3.7 we can find, for each $i$, a continuous function $\psi'_i : \mathrm{X} \to [0,1]$ such that $\psi'_i|_{\overline{\mathrm{W}}_i} = 1$ and $\psi'_i$ vanishes outside of $\mathrm{V}_i$.

Now, these conditions ensure that $\mathrm{supp}(\psi'_i) \subset \overline{\mathrm{V}}_i \subset \mathrm{U}_i$. Now, $\mathcal{W}$ is an open cover of X. Hence $\Psi' := \sum_{i=1}^{n} \psi'_i : \mathrm{X} \to [0,1]$ takes only positive values. To ensure that the sum of the functions appearing in a partition of unity sums to 1 we set

$$\psi_i := \tfrac{\psi'_i}{\Psi}.$$

□

LEMMA 2.4.7 (Refinement lemma). *Let* X *be a normal space, and* $\mathcal{U} = \{\mathrm{V}_1, \cdots, \mathrm{V}_n\}$ *be finite cover of* X *by opens. Then we can find an open cover* $\mathcal{V} = \{\mathrm{V}_1, \cdots, \mathrm{V}_n\}$ *an open cover of* X *such that* $\overline{\mathrm{V}}_i \subset \mathrm{U}_i$ *for all* $i = 1, \cdots, n$*.*

This is left to the reader or see [**Mun00**, Theorem 36.1]. The idea is to use Lemma 2.3.5.

REMARK 2.4.8 (Infinitary partitions of unity). So far we have seen partitions of unity subordinate to a finite cover. The extension to infinite cover is as follows: let $\mathcal{U} := \{U_\alpha\}$ be an open cover of X. A **partition of unity subordinate to** $\mathcal{U}$ is a family of continuous functions

$$\psi_\alpha : X \to [0,1]$$

such that (1) $\mathrm{supp}(\psi_\alpha) \subset U_\alpha$, (2) for each $x \in X$ there exists an open neighborhood U such that only finite many of $\{\psi_\alpha|_U\}$ is nonzero and (3) for each $x \in X$, the (finite) sum $\sum_\alpha \psi_\alpha = 1$. This definition of a partition of unity accommodates cases when a space does not admit an open cover with finitely many members; note that if the cover $\mathcal{U}$ has finite members then we recover the previous definition. We will discuss conditions that guarantee existence of a partition of unity after we have discussed compactness.

**2.5. Checking normality.** Let's take a moment to reflect on what makes Urysohn's lemma tick. The idea is simple: Lemma 2.3.5 allows us to exploit both the countability and the density of numbers of the form $p/2^k$ to create a suitably nice collection of closed subsets of X, interpolating between A and B. It seems that the definition of normality is actually doing the work for us. So where is the real work? Indeed, the work lies in the fact that arranging for a space to be normal can be quite quite tricky, in contrast to checking that a space is Hausdorff or regular. The following lemma is quite routine (see [**Mun00**, Theorem 31.2])

LEMMA 2.5.1. *The condition of being Hausdorff (resp. regular) is preserved under taking subspaces and products.*

PROOF. Note that spaces whose singletons are closed are stable under subspaces and products. Let's prove the result for regularity using Lemma 2.3.5. Let $A \subset X$ be a subspace, $x \in A$ and U is an open in A containing $x$. Then $U = A \cap U'$ where $U'$ is an open in X by definition of the subspace topology. By Lemma 2.3.5 there exists an open $V'$ in X such that $x \in V$ and $\overline{V}' \subset U'$. Then, $V := V' \cap A$ is an open containing $x$ such that its closure in A (computed as $\overline{V}' \cap A$) is contained in U.

Now, let $X_\alpha$ be a collection of regular spaces and consider the product $X := \prod X_\alpha$. Let $x = (x_\alpha) \in X$ and let $U = (U_\alpha)$ be a element containing $x$ (suffices to verify Lemma 2.3.5 for U a basis element); we need to find an open V of X such that $x \in V$ and $\overline{V} \subset U$. For each $\alpha$ choose $V_\alpha$ given by Lemma 2.3.5 so that $x_\alpha \in V$ and $\overline{V}_\alpha \subset U_\alpha$. Note that the product of closures is the closure of products (see [**Mun00**, Theorem 19.5]). Hence $V := \prod V_\alpha$ does the job. □

REMARK 2.5.2. Examining the proof of Lemma 2.5.1 we see the problem for verifying normality. Say Y is a subspace of X and A is a closed subset of Y. Then A *need not be closed* inside X and hence we cannot appeal to Lemma 2.3.5. Similar observation holds for products, even finite ones.

We shall see that the condition of being normal is not closed under taking subspaces or products.

EXAMPLE 2.5.3 (Uncountable products). A result of Stone (see [**Mun00**, Page 206]) proves that $(0,1)^{\omega_1}$ is not normal. This has to do with countability issues which can prevent a space from being normal (see Lemma 2.5.5 in contrast). On the other hand, we shall see from Lemma 3.3.1 that compact Hausdorff spaces are normal and Tychonoff's theorem says that arbitrary products of compact spaces are compact. Hence $[0,1]^{\omega_1}$ is normal and thus this gives an example of a normal space with a non-normal subspace.

EXAMPLE 2.5.4 (Sorgenfrey plane). Let $\mathbb{R}_\ell$ be the real line with the lower limit topology. Then it is normal: if A and B are two closed subsets, choose for each $a$ (and each $b$) an open subset $[a, x_a)$ (and $[b, x_b)$) such that $[a, x_a) \cap [a, x_b) = \emptyset$. Then $\bigcup_{a \in A} [a, x_a)$ and $\bigcup_{b \in B} [b, x_b)$ are two opens that do not intersect but contains A and B respectively. The **Sorgenfrey plane** is the space $\mathbb{R}_\ell \times \mathbb{R}_\ell$ and it is not normal as shown in [**Mun00**, Page 198]

We now explain several results which guarantee normality. First, we have that as soon as we discount infinitary phenomena (in some sense) we can guarantee the normality of a regular space.

LEMMA 2.5.5. *Every regular space with a countable basis is normal.*

PROOF. Let A and B be disjoint. We first claim that there exists countable collections of opens

$$\mathcal{U} = \{\mathrm{U}_n\} \qquad \mathcal{V} = \{\mathrm{V}_n\}$$

such that $\mathcal{U}$ is an open cover for A (in the sense that $\mathrm{A} \subset \bigcup \mathrm{U}_n$) and $\mathcal{V}$ is an open cover for B such that the closures of $\mathrm{U}_n$ (resp. of $\mathrm{V}_n$) does not intersect B (resp. A). Having such collections, we note that $\bigcup \mathrm{U}_n$ need not be disjoint from $\bigcup_n \mathrm{V}_n$, so we need to modify each the $\mathrm{U}_n$'s and $\mathrm{V}_n$'s to arrange this to happen. Indeed, for each $n$, take

$$\mathrm{U}'_n := \mathrm{U}_n \smallsetminus \bigcup_{i=1}^{n} \overline{\mathrm{V}}_n \qquad \mathrm{V}'_n := \mathrm{V}_n \smallsetminus \bigcup_{i=1}^{n} \overline{\mathrm{U}}_n.$$

□

By now we have seen that asking for a countable basis has appeared in a number of places.

DEFINITION 2.5.6. A space X is **second countable** if it has a countable basis.

We will ignore the first countability axiom because the second countability axiom is much more important. Next, we have seen that any normal space with a countable basis is metrizable.

LEMMA 2.5.7. *A metrizable space is normal.*

PROOF. For A and B close subsets, we take

$$\mathrm{U} = \{x : d(x, \mathrm{A}) < d(x, \mathrm{B})\} \qquad \mathrm{V} = \{x : d(x, \mathrm{B}) < d(x, \mathrm{A})\}$$

where $d(x, \mathrm{A}) = \inf\{d(x, a) : a \in \mathrm{C}\}$. □

The last, most important example of a normal space are compact Hausdorff spaces. This nicely motivates the transition to the next chapter.

## 3. Compactness

We finally arrive at the notion of compactness. This idea is ubiquitous throughout mathematics; let us recall the elementary definition from analysis:

DEFINITION 3.0.1. Let $\mathrm{A} \subset \mathbb{R}^n$. Then we say that A is compact if every sequence $\{x_n\} \subset \mathrm{A}$ has a convergent subsequence: that is a subsequence $\{x_m\}$ such that $x_m$ converges to a point $x$ where $x \in \mathrm{A}$ (recall that we have discussed convergence in Remark 1.1.6).

This is a very well-motivated definition and consists of two parts: firstly we don't actually care about the entire sequence $x_n$ but instead only care about convergence properties of a subsequence. Secondly, we ask that the limit of such a (sub)sequence is still contained in A: this is to say that A has "no gaps" or has "no holes". This last point is exactly the intuition that whatever notion of compactness is meant to capture. Two possible abstractions of this definition are:

DEFINITION 3.0.2. A space X is said to be **sequentially compact** if every sequence of points in X admits a convergent subsequence.

In essence, I argue that Definition 3.0.2 is the idea of compactness that we want. However, it is not a suitable definition for an arbitrary topological space for various reasons.

Example 3.0.3 (Infinite, but countable, sequences fail to detect lack of compactness). First, consider $\mathbb{N}$ the natural numbers — this is a model for a countable, totally ordered set. It should not be an example of a compact set as the natural numbers "diverge" to infinity. In other words, it is missing limit points. Guided by this intuition, let $\omega_1$, the minimal totally ordered set. It also should not be an example of a compact set: we can build an (uncountably) infinite sequence in $\omega_1$ which are unbounded.

However, $\omega_1$ is actually sequentially compact. Roughly: if $\{x_n\}$ is a sequence, take $c = \sup\{x_n\}$ then the subsequence of increasing elements converging to $c$. Here, the main point is that *any countable subset of* $\omega_1$ admits an upper bound and hence a least upper bound [**Mun00**, Theorem 10.3] (compare: any finite subset of $\mathbb{N}$ admits an upper bound).

Actually the problem is even worse.

Example 3.0.4 (Infinite, but countable, sequences fail to detect compactness). Consider $[0,1]^{\omega_1} = \prod_{\omega_1}[0,1]$ with the product topology. This is to be thought of as uncountably many elements with entries in the interval $[0,1]$. A concrete model is to take $\mathcal{P}(\mathbb{N}) = \{0,1\}^{\mathbb{N}}$ as $\omega_1$; this means we are just looking at a countable sequence of 0's and 1's and elements of $[0,1]^{\omega_1}$ (so maybe should write it as a vector $\hat{v} = (v_1, \cdots, v_n, \cdots)$) assigns to each such sequence a number between 0 and 1. One can show (as in the homework) that the sequence $f_n : \{0,1\}^{\mathbb{N}} \to [0,1]$ given by $f_n(\hat{v}) = v_n$ has no convergent subsequence. On the other hand, we want the space $[0,1]^{\omega_1}$ to be compact! The point is the space (appropriately topologized) of continuous maps into a compact space should be compact; this will be justified by Tychonoff's theorem. The problem is that as soon as the domain of this mapping space is not countable, the notion of sequential compactness fails to detect compactness.

A mental image that one should have of sequential compactness (and eventually compactness) is that of experiments: one wants to know if a space is compact so one runs experiments on it (running a countable sequence in it). However, it turns out that these experiments fail to detect compactness.

**3.1. The definition of compactness.** Without further ado, let us define the notion of compactness as is accepted these days; it has the advantage of being relatively easy to state. Let $\mathcal{U} := \{\mathrm{U}_\alpha\}_{\alpha \in \mathrm{A}}$ be a collection of open subsets of X. We say that it is a **open cover** (we sometimes just say **cover**) if $\cup_\alpha \mathrm{U}_\alpha = \mathrm{X}$. A subcover is a subset of $\mathcal{U}$ whose union still covers X

Definition 3.1.1. A space X is **compact** if every open cover has a finite subcover.

One big advantage of Definition 3.1.1 is that one can easily prove of the desiderata for the notion of compactness: that it is preserved under arbitrary continuous maps. To formulate this and other results about compactness, we introduce a construction that is ubiquitous in mathematics.

Construction 3.1.2 (Pullbacks). Suppose that we have continuous maps

$$\begin{array}{ccc} & & \mathrm{X} \\ & & \big\downarrow f \\ \mathrm{W} & \xrightarrow{g} & \mathrm{Y}. \end{array}$$

Then the **pulback** of the above diagram is the subspace of $\mathrm{X} \times \mathrm{W}$ denoted by

$$\mathrm{X} \times_{\mathrm{Y}} \mathrm{W} \subset \mathrm{X} \times \mathrm{W},$$

given by $\{(x,w) : f(x) = g(w)\}$. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{X} \times_{\mathrm{Y}} \mathrm{W} & \longrightarrow & \mathrm{X} \\ \big\downarrow & & \big\downarrow f \\ \mathrm{W} & \xrightarrow{g} & \mathrm{Y}. \end{array}$$

This square is called a **pullback square**.

Remark 3.1.3 (Universal property of pullbacks). Recall that to a continuous map to a product is $X \times Y$ is determined uniquely by continuous maps to X and to Y. The universal property for pullbacks is a generalization of this. Suppose that we are in the situation of Construction 3.1.2, then assume that we have two map $a : Z \to X, b : Z \to W$, with the additional condition that the composites $f \circ a : Z \to Y, g \circ b : Z \to Y$ agrees. Then there is a unique continuous map $Z \to X \times_Y W$. The reader is encourages to check this result in a similar way as in the the universal property of products.

Example 3.1.4. We have secretly seen pullbacks all over the place. Let $A \subset X$ be a subspace (for example, $A = \{x\}$ a singleton set) and $Y \to X$ a continuous map. Then the pullback is homeomorphic to the preimage (as a subspace of Y)

$$Y \cong A \times_X Y,$$

as one can check from the definitions. On the other hand, the pullback of $X \to * \leftarrow Y$ is the product $X \times Y$ with the product topology.

Definition 3.1.5. A continuous map $f : X \to Y$ is **closed** if for any closed subset $Z \subset X$, $f(Z) \subset Y$ is closed. We say that it is **universally closed** if for every continuous map $f : W \to Y$, the map $X \times_Y W \to W$ is closed.

Evidently, a universally closed map is closed. But the converse is not true: closed maps by themselves are usually not every interesting.

Example 3.1.6 (Closed but not universally closed). Any space has a continuous map to the point $X \to *$. It is closed because its image is $\emptyset$ (in which case, X must be actually just ) or all of $*$. Either way, it is closed. However, for any other space Y the map $Y \times X \to Y$ need not be closed. For example, let $X = Y = \mathbb{R}$. Then the projection map $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is not closed: for example the subset $\{(x, y) : xy = 1\}$ is closed in the product but its image is $\mathbb{R} \setminus \{0\}$ under the projection onto the $x$-coordinate which is not a closed subset.

On the other hand, by definition, a universally closed map is stable under pullbacks: if $f : X \to Y$ is universally closed, then for any map $W \to Y$, the map $W \times_Y X \to W$ is universally closed. Indeed, if $T \to W$ is any other continuous map, then we have a homeomorphism (compatible with the maps to T):

$$T \times_W (W \times_Y X) \cong T \times_Y X$$

Hence the map $T \times_Y X \to T$ is closed since $f$ was assumed to be universally closed.

Being universally closed can be thought of as a generalization of being a closed subspace.

Lemma 3.1.7. *Let $A \subset X$ be a closed subspace. Then the map $A \to X$ is universally closed.*

Proof. Given a map $f : W \to X$, by Example 3.1.4, $A \times_X W \cong f^{-1}(W) \to W$ identifies with the inclusion of the preimage of A under $f$. Since the preimage of closed subset is closed, $A \times_X W \to W$ is closed.

□

The following lemmas are crucial to proving basic properties about compact sets in a uniform way.

Lemma 3.1.8. *Let X be a compact space, then $X \to *$ is universally closed. In other words, for any other space W, the projection $X \times W \to W$ is closed.*

Proof. Let $K \subset X \times W$ be closed and let $p : X \times W \to W$ be the projection map. We want to prove that $W \setminus (p(K))$ is open. Let $w \in W \setminus (p(K))$ we we want to find an open neighborhood U of $w$ such that $U \cap p(K) = \emptyset$.

Let $\mathcal{U}$ be the collection of all opens V of X such that there exists an open neighborhood U of $y$ for which $(V \times U) \cap K = \emptyset$. We claim that $\mathcal{U}$ is an open cover for X. Indeed: for every $x \in X$ we have that $(y, x) \notin K$ since $p(y) \notin K$. Since K is assumed to be closed, there exists an open basis element $V \times U$ containing $(y, x)$ but not intersecting K. Now, X is compact, so we can refine $\mathcal{U}$ to a finite cover $V_1, \cdots, V_n$ of X and we can find $U_1, \cdots U_n$, opens of W containing $y$ such that $U_i \times V_i$ does not intersect K. Now, take $U := \cap U_i$ which is open since it is a finite intersection. This open set does the job.

□

Remark 3.1.9. In fact, the converse of Lemma 3.1.8 holds so a space is compact if and only if $X \to *$ is universally closed. The converse direction is much harder, but we might have occasion to discuss its proof after Tychonoff's theorem.

We will leave the next lemma an exercise.

Lemma 3.1.10. *Universally closed maps compose: if* $f : X \to Y, g : Y \to W$ *are universally closed maps, then* $g \circ f : X \to W$ *is universally closed.*

Theorem 3.1.11 (Basic properties of compact spaces). *The following statements hold:*

(1) *a closed subset of a compact set is compact;*
(2) *the image of a compact space under a continuous map is compact;*
(3) *let* Y *be a Hausdorff space and* X *compact. Let* $f : X \to Y$ *be a continuous function then* $f$ *is universally closed. In particular, a compact subspace of a Hausdorff space is closed.*

Proof. If $A \subset X$ is a closed subset and $\mathcal{U} = \{U_\alpha\}$ be an open covering of A. By definition of the subspace topology $U_\alpha = U'_\alpha \cap A$ for some $U'_\alpha \subset X$ and open subset. We take $\{U'_\alpha\} \cup \{X \smallsetminus A\}$ which is an open cover of X. By hypothesis, we can refine this a to a finite cover, which produces a finite cover of A.

(Here is another way to see the (i) if one accepts the criterion for compactness asserted in Remark 3.1.9. By Lemma 3.1.7 the map $A \to X$ is universally closed and by Lemma 3.1.8, $X \to *$ is universally closed. Therefore, by Lemma 3.1.10 the composite $A \to *$ is universally closed.)

We prove (2). Let $f : X \to Y$ be a continuous map where X is compact. Without loss of generality, we assume that $f$ is surjective. Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of Y. Then $\{f^{-1}(U_\alpha)\}$ is an open cover of X and since X is compact there is a finite subcover $\{f^{-1}(U_i)\}_{i=1,\cdots n}$. Then $\{U_i\}_{i=1,\cdots n}$ constitute an open subcover of $\mathcal{U}$.

(Again Remark 3.1.9 gives another proof of (2).)

We now prove (3). Let $h : Z \to Y$ be a continuous map; the goal is to prove that $Z \times_Y X \to Z$ is closed. We contemplate the following diagram

$$\begin{array}{ccccc} Z \times_Y X & \longrightarrow & Z \times X & \longrightarrow & Z \\ {\scriptstyle (z,x)\mapsto h(x)=f(x)}\downarrow & & \downarrow{\scriptstyle (z,x)\mapsto (h(z),f(x))} & & \\ Y & \xrightarrow{\Delta} & Y \times Y & & \end{array}$$

We claim that:

(1) the square is commutative;
(2) the top composite is the projection map $(z, x) \mapsto z$
(3) the square above is a pullback square.

If we have these claims we are done. Indeed, because Y is Hausdorff, $\Delta$ is a closed embedding (it is evidently injective and, by Hausdorfness, is a closed map), whence $\Delta$ is universally closed by Lemma 3.1.7. Hence $Z \times_Y X \to Z \times X$ is a closed map and, in fact, universally closed by the second paragraph of Example 3.1.6. By Lemma 3.1.8 the map $Z \times X \to Z$ is universally closed. The composite of universally closed maps is universally closed by Lemma 3.1.10 and therefore, $f$ is closed (in fact, universally closed). If $A \subset X$ is a compact subspace, apply the result to the map $i : A \to X$ and hence we are done.

Now to prove the claims: indeed if $(z,x) \in \mathrm{Z} \times_{\mathrm{Y}} \mathrm{X}$, going one route sends it to $(h(z), f(x))$ while going the other route sends it to $(h(x), h(x))$ but these are equal because $(z,x)$ are precisely points for which $(h(x) = f(x))$. The top composite is then simply the subspace inclusion followed by the projection and this verifies the second claim. Now, the pullback of the diagram $\mathrm{Y} \to \mathrm{Y} \times \mathrm{Y} \leftarrow \mathrm{Z} \times \mathrm{X}$ consist of the subspace of $\mathrm{Y} \times (\mathrm{Z} \times \mathrm{X})$ given by $(y,(z,x))$ such that $\Delta(y) = (h(z), f(x))$ which is to say that $(y,y) = (h(z), f(x))$ and therefore $h(z) = f(x)$. This is exactly the subspace $\mathrm{Z} \times_{\mathrm{Y}} \mathrm{X}$ as described. □

Corollary 3.1.12. *Let $f : \mathrm{X} \to \mathrm{Y}$ be a bijective continuous function where*

*(1) $\mathrm{X}$ is compact;*
*(2) $\mathrm{Y}$ is Hausdorff.*

*Then $f$ is a homeomorphism.*

Proof. Since $f$ is closed (and hence takes a closed set to a closed set) by Theorem 3.1.11(3), we have that $f^{-1}$ must be continuous since the preimage of a closed set is a closed set. □

**3.2. Application: extreme value theorem and revisiting connectedness.** Now that we have seen some results about compact spaces, we would like to give examples of them. In particular, we would like to prove the Heine-Borel theorem that says a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded, as well as its generalizations. One of the key points of the Heine-Borel theorem is that it makes meaning out of the following result

Theorem 3.2.1 (Extreme value theorem). *Let $\mathrm{X} \to \mathrm{Y}$ be a continuous map where $\mathrm{X}$ is compact and $\mathrm{Y}$ is a totally ordered set with the order topology. Then, there exists $c, d \in \mathrm{X}$ such that for all $x \in \mathrm{X}$ we have that $f(c) \leqslant f(x) \leqslant f(d)$. In other words, $f$ attains a minimum and a maximum value.*

Proof. By Theorem 3.1.11(3), $f(\mathrm{X})$ is compact. It suffices to prove that $f(\mathrm{X})$ contains a smallest element and a largest element. Assume that $f(\mathrm{X})$ does not have a largest element, the consider the collection of opens $\{\{(-\infty, y) : y \in f(\mathrm{X})\}\}_{y \in f(\mathrm{X})}$. It covers $f(\mathrm{X})$. Now $f(\mathrm{X})$ is compact, hence we can refine the above to a finite subcover $(-\infty, y_i)$ for $i = 1, \cdots, n$. Let $y_k$ be the largest element among $y_1, \cdots y_n$. Then $y_k$ cannot in $f(\mathrm{X})$ since we had assumed that it does not have a largest element. But this means that this finite subcover is not a cover, contradicting compactness of $f(\mathrm{X})$. The same argument gives the result about smallest element.

□

To recover the usual extreme value theorem, we must prove:

Theorem 3.2.2. *The interval $[0,1]$ is compact.*

We will prove Theorem 3.2.2 later using Tychonoff's theorem. Having this we can prove

Theorem 3.2.3. *The interval $[0,1]$ is connected. In particular, $[a,b]$ is connected for all $a \leqslant b$.*

Proof. Assume that $[0,1]$ has a separation U and V. Then we can also express $[0,1]$ as $\mathrm{A} \sqcup \mathrm{B}$ where A and B are nonempty, closed, disjoint subsets. By Theorem 3.2.2 and Theorem 3.1.11(1), A and B are compact. By Lemma 3.2.4, $\mathrm{A} \times \mathrm{B}$ is compact. It inherits a metric from $[0,1] \times [0,1]$ which defines a distance function

$$d : \mathrm{A} \times \mathrm{B} \to \mathbb{R}.$$

The distance function for any metric space is continuous (see Lemma 3.2.5 below) and therefore, Theorem 3.2.1 tells us that it attains a minimum. If this value is zero, then $\mathrm{A} \cap \mathrm{B} \neq \emptyset$ which is a contradiction. If not, take any point $c \in (a,b)$. Then this point is neither in A nor B because its distance from $a$ and $b$ will be smaller than the minimum, contradicting $[0,1] = \mathrm{A} \sqcup \mathrm{B}$. □

Lemma 3.2.4. *A finite product of compact spaces are compact.*

Proof. We shall prove Tychonoff's theorem which says that this is true for any product and so we postpone this. □

Lemma 3.2.5. *For any metric space* $(\mathrm{X}, d)$*, the distance function* $d : \mathrm{X} \times \mathrm{X} \to \mathbb{R}$ *is continuous.*

Proof. This a standard result in analysis; as a hint: fix $x_0$ and consider $f : \mathrm{X} \to \mathbb{R}$ defined as $f(x) = d(x, x_0)$. Then the triangle inequality gives $|f(x) - f(y)| \leqslant d(x, y)$; one then just applies the $\epsilon$-$\delta$ definition of continuity. □

Proof of Theorem 1.0.6 for the reals. Recall that a set $\mathrm{L} \subset \mathbb{R}$ is said to be **convex** if for any $a, b \in \mathrm{L}$ with $a < b$ we have that $[a, b] \subset \mathrm{L}$. We claim that any convex set L of the real line is connected and this will be enough to prove that rays and the real line itself is connected. Indeed, if L has a separation $\mathrm{U}, \mathrm{V}$. Then choose $a \in \mathrm{U}$ and $b \in \mathrm{V}$ and assume, without loss of generality, that $a < b$. Then the interval $[a, b] \subset \mathrm{L}$ by hypothesis. Therefore, $\mathrm{U} \cap [a, b]$ and $\mathrm{V} \cap [a, b]$ constitute a separation of the interval, contradicting Theorem 3.2.3

□

Theorem 3.2.6 (Heine-Borel). *A subset* $\mathrm{A} \subset \mathbb{R}^n$ *is compact if and only if it is closed and bounded (with respect to, say, the Euclidean metric).*

Proof. Let A be compact. According to Theorem 3.1.11(3), it must be closed. We can consider $\{\mathrm{B}^\circ_\epsilon(x)\}_{x \in \mathrm{A}}$, the collection of open balls around every point of A. It is an open cover of A and so, since A is compact, it must have a finite subcover. Therefore A is bounded.

Now, let us assume that A is closed and bounded. Then there exists some large enough N such that $\mathrm{A} \subset [-\mathrm{N}, \mathrm{N}]^{\times n}$ and is a closed subset. Theorem 3.2.2 implies that $[-\mathrm{N}, \mathrm{N}]$ is compact and Lemma 3.2.4 says that $[-\mathrm{N}, \mathrm{N}]^{\times n}$ is compact. Since A is a closed subset of a compact set, it must be compact. □

Remark 3.2.7 (Connectedness of linear continua). The argument above does not prove Theorem 1.0.6: there are linear continua which are connected but not metrizable. For example, the long line. However, the proof is actually quite similar for linear continua if you examine [**Mun00**, Theorem 24.1] if you think about it carefully.

**3.3. Compactness and separation axioms, CW complexes.** We have seen that Hausdorffness and compactness are conditions that go well with each other. In fact, it is even better:

Lemma 3.3.1. *Compact Hausdorff spaces are normal.*

Proof. We first claim that X must be regular. Let $x \in \mathrm{X}$ and $\mathrm{B} \subset \mathrm{X}$ where B is closed and does not contain $x$. Then we know by Theorem 3.1.11(1) that B is compact because every closed subset of a compact must be compact. By Hausdorffness, for each $b \in \mathrm{B}$ we can find $\mathrm{U}_b$ containing $b$ and $\mathrm{V}_b$ containing $x$ such that $\mathrm{V}_b$ and $\mathrm{U}_b$ are disjoint; note that $\{\mathrm{U}_b\}$ is an open cover of B. By compactness of B we can refine $\{\mathrm{U}_b\}$ to a finite subcover $\mathrm{U}_1, \cdots \mathrm{U}_n$. Then $\mathrm{U}_1 \cap \cdots \cap \mathrm{U}_n$ and $\mathrm{V}_1 \cap \cdots \cap \mathrm{V}_n$ are open subsets which contains B and $x$ respectively and are disjoint. Now if we are given $\mathrm{A}, \mathrm{B}$ be two closed subsets of a compact Hausdorff space X we can perform the same maneuver as above utilizing the compactness of A. Details are left to the reader.

□

This yields, thanks to Urysoh's metrization theorem, the following result.

Corollary 3.3.2. *Second countable, compact, Hausdorff spaces are metrizable.*

Remark 3.3.3 (Second countability versus compactness). We remark that second countability (which asserts the existence of a countable basis) is disjoint notion from being compact — even though it seems that both asserts something about finiteness of open sets. For example, by Lemma 2.5.1 and Tychonoff's theorem any product of compact Hausdorff spaces remain compact Hausdorff. So something like the uncountable product of the two-point set with the discrete topology is a compact Hausdorff space. On the other hand, it is not second countable. However, any compact *metric space* turns out to be second countable (given as an exercise).

Since compact Hausdorff spaces are normal, we have also shown.

Corollary 3.3.4. *Any compact Hausdorff space admits a finite partition of unity.*

As in previous notions, we ask the question: where does one go about finding compact Hausdorff spaces? The key tool for answering this question will be Tychonoff's theorem which is the subject of the next chapter. For now, we will prove that CW complexes, with finitely many cells, are actually compact and Hausdorff.

Theorem 3.3.5. *Any CW complex with finitely many cells is compact and Hausdorff. If it has infinitely many cells, then it is Hausdorff and, in fact, normal.*

Theorem 3.3.5 tells us that any space constructed by attaching cells is really quite nice. To prove Theorem 3.3.5 recall that CW complexes are built inductively using cells. If $\mathrm{X}^n$ has been built and is compact and Hausdorff, then we are given a map $\varphi : \mathrm{S}^n \to \mathrm{X}$ and contemplate the quotient map

$$p : \mathrm{X}^n \sqcup \mathrm{D}^{n+1} \to \mathrm{X}^n \cup_\varphi \mathrm{D}^{n+1}.$$

By assumption X is compact. The unit disk $\mathrm{D}^{n+1}$ is homeomorphic to $[0,1]^{\times n+1}$ as one can easily see. Therefore it is compact. Since $p$ is surjective and continuous we have that $\mathrm{X}^n \cup_\varphi \mathrm{D}^{n+1}$ is compact. The hard part is to verify Hausdorfness. However, to prove Hausdorfness we need compactness as well; secretly we are verifying normality.

To proceed, let us prove something more general. We introduce the "dual" construction to a pullback.

Construction 3.3.6 (Pushouts). Suppose that we have continuous maps

$$\begin{array}{ccc} \mathrm{Y} & \xrightarrow{f} & \mathrm{X} \\ \downarrow{\scriptstyle g} & & \\ \mathrm{W} & & \end{array} .$$

Then the **pushout** of the above diagram is the quotient

$$\mathrm{X} \sqcup_\mathrm{Y} \mathrm{W} := \mathrm{X} \sqcup \mathrm{W}/\sim$$

where the equivalence relation is given by $f(y) \simeq g(y)$ for all $y \in \mathrm{Y}$. It admits quotient maps $p \sqcup q : \mathrm{X} \sqcup \mathrm{W} \to \mathrm{X} \sqcup_\mathrm{Y} \mathrm{W}$ such that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Y} & \xrightarrow{f} & \mathrm{X} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle p} \\ \mathrm{W} & \xrightarrow{q} & \mathrm{X} \cup_\mathrm{Y} \mathrm{W}. \end{array}$$

This square is called a **pushout square**.

Remark 3.3.7 (Universal property of pushouts). This remark is dual to Remark 3.1.3. A continuous map $\mathrm{X} \sqcup \mathrm{Y} \to \mathrm{W}$ is determined uniquely by continuous maps $\mathrm{X}, \mathrm{Y} \to \mathrm{W}$. The universal property of pushouts is a generalization of this. Suppose that we are in the situation of Construction 3.1.2, then assume that we have two map $a : \mathrm{X} \to \mathrm{Z}, b : \mathrm{W} \to \mathrm{Z}$, with the additional condition that the composites $a \circ f : \mathrm{Y} \to \mathrm{Z}, b \circ g : \mathrm{Y} \to \mathrm{Z}$ agrees. Then there is a unique continuous map $\mathrm{X} \cup_\mathrm{Y} \to \mathrm{Z}$. The reader is encourages to check this result in a dual way to the universal property of products.

Remark 3.3.8. Cell attachment is an instance of the pushout construction where $g$ is the inclusion map $\mathrm{S}^n \to \mathrm{D}^{n+1}$ and $f$ is the map $\varphi : \mathrm{S}^n \to \mathrm{D}^{n+1}$. In most examples, the map $f$ or $g$ will be inclusion of a closed subspace. We will see why this is thanks to the following result.

In most cases, the map $\mathrm{Y} \to \mathrm{X}$ is the inclusion of a closed subspace, just as in the case of CW complexes. This gives us a nicer description of the pushout.

LEMMA 3.3.9. *In Construction 3.1.2, if* $\mathrm{Y} \to \mathrm{X}$ *is an inclusion of a closed subspace, then the map* $\mathrm{W} \to \mathrm{X} \cup_{\mathrm{Y}} \mathrm{W}$ *is also the inclusion of a closed subspace.*

PROOF. The equivalence classes in $\mathrm{X} \sqcup \mathrm{Y}$ is given by singletons $\{x\}$ if $x \in \mathrm{X} \setminus \mathrm{Y}$ or $\{w\} \sqcup g^{-1}(w)$. Therefore, as a set, $\mathrm{X} \cup_{\varphi} \mathrm{Y}$ is the disjoint union of $\mathrm{X} \setminus \mathrm{Y}$ and W (but not topologized as a disjoint union). Therefore, the map $\mathrm{W} \to \mathrm{X} \cup_{\mathrm{Y}} \mathrm{W}$ is injective. That the map is closed follows from the way that $\mathrm{X} \cup_{\mathrm{Y}} \mathrm{W}$ is topologized: a subset $\mathrm{Z} \subset \mathrm{X} \cup_{\mathrm{Y}} \mathrm{W}$ is closed if and only if $p^{-1}(\mathrm{Z})$ and $q^{-1}(\mathrm{W})$ are closed. $\square$

LEMMA 3.3.10. *Let* $\mathrm{X}, \mathrm{Y}$ *be compact Hausdorff spaces and let* $\mathrm{A} \subset \mathrm{X}$ *be a closed subspace and* $\varphi : \mathrm{A} \to \mathrm{Y}$ *be a continuous map. Then the space* $\mathrm{X} \cup_{\varphi} \mathrm{Y}$ *is compact Hausdorff.*

PROOF. We have already seen that $\mathrm{X} \cup_{\varphi} \mathrm{Y}$ is compact since it is the quotient of a compact space. It remains to prove Hausdorfness. In particular we will prove normality. Recall that Tietze's extension theorem and Urysohn's lemma are equivalent. In particular a space W is normal if and only if for each closed subset $\mathrm{Z} \subset \mathrm{W}$ and any continuous map $f : \mathrm{Z} \to \mathbb{R}$, there exists a continuous extension $\widetilde{f} : \mathrm{W} \to \mathbb{R}$. Furthermore, we note that compact Hausdorff spaces are normal as proved in Lemma 3.3.1.

So we are given $\mathrm{Z} \subset \mathrm{X} \cup_{\varphi} \mathrm{Y}$ a closed subset and a continuous map $t : \mathrm{Z} \to \mathbb{R}$. By Remark 3.3.7 to construct a map out of the pushout, it suffices to construct maps $f : \mathrm{X} \to \mathbb{R}, g : \mathrm{Y} \to \mathbb{R}$ that agrees on A. Our diagram looks like:

$$\begin{array}{ccc} \mathrm{A} & \xhookrightarrow{\ i\ } & \mathrm{X} \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle p} \\ \mathrm{Y} & \xhookrightarrow{\ q\ } & \mathrm{X} \cup_{\varphi} \mathrm{Y}. \end{array}$$

First, by Lemma 3.3.9, $q$ is the inclusion of a closed subspace. Hence, $\mathrm{Z} \cap \mathrm{Y}$ is closed and we have a map $t|_{\mathrm{Z} \cap \mathrm{Y}} : \mathrm{Z} \cap \mathrm{Y} \hookrightarrow \mathrm{Z} \xrightarrow{t} \mathbb{R}$. Since Y is normal, this extends to a map out of Y by the Tietze extension theorem, call it $g : \mathrm{Y} \to \mathbb{R}$.

Now, consider $p^{-1}(\mathrm{Z}) \subset \mathrm{X}$. Now, the maps $\mathrm{A} \xrightarrow{\varphi} \mathrm{Y} \xrightarrow{g} \mathbb{R}$ and $p^{-1}(\mathrm{Z}) \xrightarrow{p|_{p^{-1}(\mathrm{Z})}} \mathrm{Z} \xrightarrow{t} \mathbb{R}$ agrees on $\mathrm{A} \cap p^{-1}(\mathrm{Z})$ because they factor through the map $t : \mathrm{Z} \to \mathbb{R}$. Therefore, this defines a map $\mathrm{A} \cup p^{-1}(\mathrm{Z}) \to \mathbb{R}$. Now, $\mathrm{A} \cup p^{-1}(\mathrm{Z}) \subset \mathrm{X}$ is closed and X is normal, hence we get a map $f : \mathrm{X} \to \mathbb{R}$. By construction the maps $f$ and $g$ agree on A and therefore, by the universal property of pushouts, we get a map $\widetilde{t} : \mathrm{X} \cup_{\varphi} \mathrm{Y} \to \mathbb{R}$ extending $t$. $\square$

PROOF OF THEOREM 3.3.5. We will prove the assertion about finite CW complexes. Since finitely many points, which constitute $\mathrm{X}^0$, is compact and Hausdorff we have the base case of an induction. By induction, we may assume that $\mathrm{X}^n$ is compact Hausdorff. But $\mathrm{X}^{n+1}$ is built from $\mathrm{X}^n$ by the taking pushouts along $\mathrm{S}^n \to \mathrm{D}^{n+1}$ which satisfies the hypothesis of Lemma 3.3.10. Therefore $\mathrm{X}^{n+1}$ remains compact Hausdorff.

$\square$

CHAPTER 4

# Tychonoff's Theorem and further properties of (locally) compact Hausdorff spaces

The goal of this chapter is to convince you that being a compact Hausdorff spaces is a property that covers a lot of interesting spaces (of a wide variety). In particular, we venture beyond the realm of spaces that can be "visualized" (like manifolds etc.) but are still pervasive throughout mathematics.

Furthermore, we will explore the idea of *compactifications*: if $\mathrm{X}$ is not necessarily compact then we want to find the "best compact approximation" to $\mathrm{X}$. There are two kinds of compactifications that we will explore. The first is a sort of "smallest" compactification of $\mathrm{X}$ called the one-point compactification

$$\mathrm{X} \hookrightarrow \mathrm{X}^{+}.$$

One imagine this as adjoining a point "at $\infty$" which behaves like a limit point of all sequences. It is often quite intuitive and sometimes possible to describe explicitly. However, one-point compactifications do not enjoy universal properties: to map out of $\mathrm{X}^{+}$ given a map $\mathrm{X} \to \mathrm{Y}$, one needs to specificy where the point at $\infty$-goes. This leads to a somewhat more exotic compactification called the Stone-Čech compactification:

$$\mathrm{X} \hookrightarrow \beta(\mathrm{X}).$$

It is, in a sense, the largest possible compactification of $\mathrm{X}$ that one can build. So large that it is forced to have a universal property.

The flagship theorem of this part of the course, and the key point behind the existence of Stone-Čech compactifications, is Tychonoff's theorem:

Theorem 0.0.1 (Tychonoff). *Arbitrary products of compact spaces are compact. More precisely, if* $\mathrm{A}$ *is a nonempty set and* $\{\mathrm{X}_\alpha\}_{\alpha\in\mathrm{A}}$ *a collection of compact spaces. Then* $\prod_\alpha \mathrm{X}_\alpha$ *is compact.*

The first order of business is to prove Tychonoff's theorem.

## 1. Nets and filters

Filters are rather abstract objects and can look unmotivated. To get an idea of what they are and what they are good for we first discuss a generalization of sequences.

### 1.1. A quick introduction to nets.

Definition 1.1.1. A **directed set** is a set $\mathrm{I}$ with a binary relation $\leqslant$ such that:

(1) it is nonempty
(2) it is reflexive: $\alpha \leqslant \alpha$;
(3) it is transitive: if $\alpha \leqslant \beta$ and $\beta \leqslant \gamma$ then $\alpha \leqslant \gamma$;
(4) for any two $\alpha, \beta \in \mathrm{I}$ there exists a $\gamma$ such that $\alpha \leqslant \gamma$ and $\alpha \leqslant \gamma$.

We write $(\mathrm{I}, \leqslant)$ to indicate a directed set.

Example 1.1.2 (Natural numbers). The simplest example of a directed set is the natural numbers with $\leqslant$ interpreted as usual; we write this as $(\mathbb{N}, \leqslant)$. Of course $(\mathbb{Z}, \leqslant)$ is also a directed set.

EXAMPLE 1.1.3 (Real numbers). Totally ordered sets are examples of directed sets and so the real numbers with the usual $\leqslant$ relation is also a directed set: $(\mathbb{R}, \leqslant)$.

EXAMPLE 1.1.4 (The directed set of opens). Let X be a space with $\mathcal{T}$ its topology. Then we declare $U \leqslant V$ if and only if $U \subset V$. We claim that this is a directed set. Indeed, since $\emptyset$ and X are in $\mathcal{T}$ it is not empty. Evidently, $U \subset U$. Furthermore if $U \subset V, V \subset W$ then $U \subset W$. Now: if $U, V \subset \mathcal{T}$ then setting $U \cup V$ we have that it is open and $U \subset U \cup V$ and $V \subset U \cup V$. This is the first genuine example of a directed set that's not totally ordered: in general we can't really compare two open sets because they may not intersect.

EXAMPLE 1.1.5 (Neighborhoods). The following is an important example of a directed set: fix $x \in X$ and consider

$$\mathcal{N}_x := \{U \subset X : U \text{ is open and } x \in X\}.$$

We claim that this is a directed set under the *superset* relation: this means that $U \leqslant V$ if and only if $V \subset U$ (this is not a typo!). We think of V as *refining* U because it is closer to $x$ than U.

(1) X is an open set in $\mathcal{N}_x$;
(2) reflexivity is evident;
(3) transitivity is evident too;
(4) this is the key point: if $U, V \in \mathcal{N}_x$ then $U \cap V$ is an open set which still contains $x$ and $U \cap V \subset U, V$.

So the idea of a directed set is that we have a set with an ordering but we may not be able to compare two points. However, axiom 4 of Definition 1.1.1 tells us that any two sets have a common refinement. Remember that one of the things we wish to do is to invent a notion of limits that rectify sequential compactness. Since limits only really care about the "tail end" of a sequence, this axiom is very natural to ask for.

DEFINITION 1.1.6 (Nets). A **net** is a function $\varphi : I \to X$ where I is a directed set. In this case, we call the net $\varphi$ an **I-indexed net**.

The usual notion of a sequence is a $\mathbb{N}$-indexed net.

EXAMPLE 1.1.7. The language of nets allows us to think of

$$\mathrm{id} : \mathbb{R} \to \mathbb{R}$$

as a "generalized sequence" of sort since a net is really such a thing. This is an example of an $\mathbb{R}$-indexed net.

EXAMPLE 1.1.8. A more interesting example is the following. Let's construct the following:

$$\mathcal{N}_x \to X \qquad U \mapsto x_U$$

where $x_U \in U$ is some element in U. This is an example of a net that is more sophisticated than a sequence and is a $\mathcal{N}_x$-indexed net.

Let us recall the following, often dubbed the "sequence lemma" (for example see [**Mun00**, Lemma 21.2]).

LEMMA 1.1.9. *Let* $A \subset X$ *be a subset. If there is a sequence of points in* A *converging to* $x \in X$ *then* $x \in \overline{A}$.

PROOF. If $x_n \to x$ then every neighborhood of $x$ will intersect A. □

REMARK 1.1.10. The converse to Lemma 1.1.9 states that if $x \in \overline{A}$ then there is a sequence of points in A converging to $x$. Intuitively, this means that there is a "witness" to $x$ being in the closure of A. We claim that if X is metrizable then this holds. Indeed for each $n \in \mathbb{N}$ we choose $x_n$ to be some element in $B_{1/n}(x)$, the ball of radius $1/n$ centered at $x$. Then, by design, $x_n$ will converge to $x$.

Example 1.1.11 (Failure of the converse of the sequence lemma). Let J be an uncountable set. Then $\prod_{\mathrm{J}} \mathbb{R}$ does not satisfy the sequence lemma. We consider $\mathrm{A} \subset \prod_{\mathrm{J}} \mathbb{R}$ to be those sequences such that $x_\alpha = 1$ except for finitely many $\alpha$'s. Then $0 = (0, 0, \cdots, 0, \cdots)$ is in the closure, but there is no sequence converging to it.

We use Lemma 2.1.5 to see this. Let $\mathrm{U} \subset \prod_{\mathrm{J}} \mathbb{R}$ be an open neighborhood of 0. Then U has all but finitely many coordinates, say $\alpha_1, \cdots, \alpha_k$ being all of $\mathbb{R}$. Now $\mathrm{A} \cap \mathrm{U}$ is nonempty. Indeed the element where $x_\alpha = 0$ only for $\alpha = \alpha_1, \cdots, \alpha_k$ and is 1 otherwise is in this intersection. The same statement also holds for any infinite set J so we have not used uncountability.

The problem is that no sequence converges to 0. The point is that for $x_n \to 0$, we must have that $x_n$ has more 0's in its coordinates than 1's. But these 0's can only happen at most countably many times and so can't converge rapidly enough to 0. See [**Mun00**, Page 133] for details.

The idea of a net rectifies the failure of the sequence lemma.

Definition 1.1.12. Let X be a space and $\varphi : \mathrm{I} \to \mathrm{X}$ be a net.

1. write $\mathrm{T}_\alpha := \{\varphi(\beta) : \beta \geqslant \alpha\} \subset \mathrm{X}$;
2. we say that $\varphi$ **coverges to** $x$ and write $\varphi \to x$ if for each open $\mathrm{U} \subset \mathrm{X}$ containing $x$ there exists $\alpha_{\mathrm{U}} \in \mathrm{I}$ such that
$$\mathrm{T}_{\alpha_{\mathrm{U}}} \subset \mathrm{U}.$$
3. In this case, we say that $x$ **is the limit point of the net** $\varphi$.
4. given a subset $\mathrm{A} \subset \mathrm{X}$ we say that $\varphi : \mathrm{I} \to \mathrm{X}$ is **eventually in** A if A contains a subset of the form $\mathrm{T}_\alpha$ for some $\alpha \in \mathrm{A}$.

Lemma 1.1.13. *Let* $\mathrm{A} \subset \mathrm{X}$ *be a subset. Then the following are equivalent:*

*(1)* $x \in \overline{\mathrm{A}}$*;*
*(2) there is a net* $\varphi : \mathrm{I} \to \mathrm{X}$ *such that* $\varphi \to x$*.*

Proof. That (2) $\Rightarrow$ (1) has the same proof as in the case of sequences.

Suppose that $x \in \overline{\mathrm{A}}$. We construct a net converging to $x$. Consider $\mathcal{N}_x$ as in Example 1.1.8 which is a directed set. For each $\mathrm{U} \in \mathcal{N}_x$ we have that $\mathrm{U} \cap \mathrm{A} \neq \emptyset$ by Lemma 2.1.5. Hence we may choose, for each $\mathrm{U} \in \mathcal{N}_x$ an element $x_{\mathrm{U}} \in \mathrm{U}$. We can then construct the net
$$\varphi : \mathcal{N}_x \to \mathrm{X} \qquad \mathrm{U} \mapsto x_{\mathrm{U}}.$$
We claim that $\varphi \to x$. Indeed, let $x \in \mathrm{U}$ an open neighborhood. For any $\mathrm{V} \subset \mathrm{U}$ which is in $\mathcal{N}_x$, i.e., $\mathrm{U} \leqslant \mathrm{V}$, we have that $\varphi(\mathrm{V}) = x_{\mathrm{V}} \in \mathrm{V} \subset \mathrm{U}$. Therefore, $\mathrm{T}_{x_{\mathrm{U}}} \subset \mathrm{U}$ as desired. ☐

Before we move on to filters, we remark on a key feature that nets and sequences share: that to determine convergence one can ignore finitely many elements in the sequence.

Definition 1.1.14 (Cofinal nets). Let $\varphi : \mathrm{I} \to \mathrm{X}, \psi : \mathrm{J} \to \mathrm{X}$ be two nets. Then we say that J **is cofinal in** I if there exists a map of sets $h : \mathrm{J} \to \mathrm{I}$ rendering the diagram
$$\begin{array}{ccc} \mathrm{J} & \xrightarrow{h} & \mathrm{I} \\ & \searrow \quad \swarrow & \\ & \mathrm{X} & \end{array} .$$
such that:

1. $h$ preserves the order relation: if $j \leqslant j' \in \mathrm{J}$ then $h(j) \leqslant h(j')$ in I;
2. for each $i \in \mathrm{I}$ there exists a $j \in \mathrm{J}$ such that $i \leqslant h(j)$.

The following result is easy to see from definitions.

Lemma 1.1.15 (Cofinal nets and limits). *If* $\mathrm{I} \to x$ *and* J *is cofinal in* I*, then* $\mathrm{J} \to x$*.*

**1.2. Filters and ultrafilters.** Having acquainted ourselves with the idea of a net, we now study a more general concept: that of an ultrafilter. We will see that every net defines an ultrafilter but there are ultrafilters that do not come from a net. We recommend some notes of Katchourian [] for further details and out of which some of the proofs here borrowed.

Definition 1.2.1. Let X be a set. A **filter** is a subset $\mathrm{F} \subset \mathcal{P}(\mathrm{X})$ such that:

(1) $\emptyset \notin \mathrm{F}$;
(2) it is closed under finite intersections;
(3) it is upward closed: if $\mathrm{Z} \in \mathrm{F}$ and $\mathrm{Z} \subset \mathrm{Z}'$ then $\mathrm{Z}' \in \mathrm{F}$.

A filter F is an **ultrafilter** if it is not properly contained in any other filter of X, i.e., if $\mathrm{F}'$ is a filter such that $\mathrm{F} \subset \mathrm{F}'$ then $\mathrm{F} = \mathrm{F}'$. A **subfilter** of F is a subset $\mathrm{F}' \subset \mathrm{F}$ which is itself a filter.

Let us give examples of filters:

Example 1.2.2 (Trivial filter). Let X be a nonempty set, then the singleton $\{\mathrm{X}\}$ in the power set $\mathcal{P}(\mathrm{X})$ is a filter.

Example 1.2.3 (Neighborhood filter). Consider $\mathcal{N}_x$ as above. Then it is a filter. It is evidently nonempty, closed under finite intersections and upward closed. There is something slightly better though. Consider

$$\mathcal{F}_x := \{\mathrm{A} \subset \mathrm{X} : \text{there exists } \mathrm{U} \in \mathcal{N}_x \text{ such that } \mathrm{U} \subset \mathrm{A}\}.$$

This is again a filter which is larger than $\mathcal{N}_x$. Yet it is still not maximal as Example 1.2.4 shows.

Example 1.2.4 (Principle ultrafilters). Consider $\mathcal{F}'_x := \{\mathrm{A} \subset \mathrm{X} : x \in \mathrm{A}\}$. This is larger than $\mathcal{F}_x$ because we no longer require open-ness of A; so for example the singleton set $\{x\} \in \mathcal{F}'_x$ but often it is not open. We claim that it is maximal. Indeed: assume that $\mathcal{F}'_x \subset \mathrm{F}$. If the containment is proper, take $\mathrm{A} \in \mathcal{F} \setminus \mathcal{N}_x$. Then A does not contain $\{x\}$. But then $\{x\} \in \mathcal{N}'_x$ and $\{x\} \cap \mathrm{A} = \emptyset$ which contradicts the first axiom of a filter. A filter of this form is called a **principal ultrafilter**.

The following result characterizes ultrafilters.

Proposition 1.2.5. *Let* X *be a set and* F *a filter on* X. *Then the following are equivalent:*

*(1)* F *is an ultrafilter*
*(2) for any* $\mathrm{A} \subset \mathrm{X}$ *either* $\mathrm{A} \in \mathrm{F}$ *or* $\mathrm{X} \setminus \mathrm{A} \in \mathrm{F}$.

Proof. Let F be an ultrafilter and asssume that it does not contain A. We claim that $\mathrm{X} \setminus \mathrm{A} \in \mathrm{F}$. We note that $\mathrm{F} \subset \mathrm{F} \cup \{\mathrm{A}\}$ which means, by maximality of F that $\mathrm{F} \cup \{\mathrm{A}\}$ is not a filter. Furthermore, it cannot be closed under finite intersection: otherwise it would generate a filter larger than F. Hence, for some $\mathrm{B} \in \mathrm{F}$ we have that $\mathrm{A} \cap \mathrm{B} = \emptyset$. But this means that $\mathrm{B} \subset \mathrm{X} \setminus \mathrm{A}$ and thus $\mathrm{X} \setminus \mathrm{A}$ must be in F since F is upward closed.

Assume that F satisfies property (2). For contradiction, assume that F is not maximal. In particular, there is a set A which is not in F such that $\mathrm{F} \bigcup \{\mathrm{A}\}$ is contained in a larger filter $\mathrm{F}'$. By property (2), this means that $\mathrm{X} \setminus \mathrm{A} \in \mathrm{F}$. But this means $\mathrm{X} \setminus \mathrm{A}$ and A are in $\mathrm{F}'$. Since $\mathrm{F}'$ is a filter, this means that $\mathrm{A} \cap (\mathrm{X} \setminus \mathrm{A}) = \emptyset$ is in $\mathrm{F}'$ which cannot happen. □

Example 1.2.6 (Filters from nets). Let $\varphi : \mathrm{I} \to \mathrm{X}$ be a net. Consider

$$\mathrm{F}_\varphi := \{\mathrm{A} \subset \mathrm{X} : \varphi \text{ eventually in } \mathrm{A}\}$$

Then we claim that $\mathrm{F}_\varphi$ is a filter. Clearly $\emptyset \notin \mathrm{F}_\varphi$ and $\mathrm{F}_\varphi$ is upward closed. Let us prove the finite intersection property. If $\mathrm{A}, \mathrm{B} \in \mathrm{F}_\varphi$, then there are sets $\mathrm{T}_\alpha, \mathrm{T}_\beta$ contained in A and B respectively. Since I is directed, there exists an $\gamma$ such that $\alpha, \beta \leqslant \gamma$. Therefore, $\mathrm{T}_\gamma \subset \mathrm{A} \cap \mathrm{B}$.

In fact, one can go from filters to nets. However, we will not pursue this direction.

Here's the last key point about ultrafilters for which we will need Zorn's lemma. It will be needed in Proposition 1.3.3 when we prove an ultrafilter characterization of compactness.

LEMMA 1.2.7. *Every filter is contained in an ultrafilter.*

PROOF. Let F be a fixed filter on X. Consider

$$\mathcal{F} := \{G : G \text{ is a filter on } X \text{ and } F \subset G\}.$$

We verify the hypotheses of Zorn's lemma. First, $\mathcal{F}$ is nonempty and is a poset, ordered by inclusion. Let $\mathcal{G}$ be a totally ordered subset of $\mathcal{F}$; we claim that $H := \bigcup_{G \in \mathcal{G}} G$ is a filter that contains F. This will ensure that H is an upper bound for every element in $\mathcal{G}$.

Clearly, H does not contain the empty set as it is a union of filters. Let $A, B \in H$. We claim that $A \cap B \in H$. This means that there exists $G, G'$ filters, containing F, such that $A \in G$ and $B \in G'$. Since $\mathcal{G}$ is totally ordered, we assume that $G \subset G'$. Then $A, B \in G'$ which means that $A \cap B \in G' \subset H$. In the same way, the reader can check upward closedness. □

**1.3. Convergence via filters.** We now discuss convergence from the point of view of filters.

DEFINITION 1.3.1. Let X be a space and F be a filter. We say that F **converges to** $x$ and write $F \to x$ if $\mathcal{F}_x \subset \mathcal{F}$. We say that F is a **convergent filter** if there exists a point $x \in X$ such that F converges to $x$.

The following lemma is an analog of Lemma 1.1.13; the proof will be left to the reader and is a reformulation of the same proof.

LEMMA 1.3.2. *Let $A \subset X$ be a subset. Then the following are equivalent:*

*(1) $x \in \overline{A}$;*
*(2) there is a filter F such that F converges to $x$.*

We now get straight to the point: proving Tychonoff's theorem via ultrafilters. The following reformulates compactness in terms of ultrafilters; it renders our dream that compactness can be detected by convergent sequences true as long as we replace sequences with the ultrafilters.

PROPOSITION 1.3.3 (Characterization of compactness via ultrafilters). *A space X is compact if and only if every ultrafilter converges.*

To prove Proposition 1.3.3 we will need the following two lemmas:

LEMMA 1.3.4. *Let X be a set and F be an ultrafilter on X. Let $X = X_1 \cup X_2 \cdots \cup X_n$. Then there exists a $k$ such that $X_k \in F$.*

PROOF. Assume not, then for each $k$ we have that $Y_k := X \setminus X_k$ must be in F by the ultrafilter property, Proposition 1.2.5. Since filters are closed under finite intersections, we have that $\cap Y_k \in F$. But since X is a union of the $X_k$'s this intersection is empty, contradicting that F is a filter. □

The formulate the next lemma, we say that a collection of subsets of X, say $\mathcal{A}$ has the **finite intersection property** if every finite collection of $\{A_1, \cdots, A_n\} \subset \mathcal{A}$ we have that $\bigcap_{i=1}^{n} A_i \neq \emptyset$

LEMMA 1.3.5. *A space X is compact if and only if for every collection $\mathcal{A}$ of closed subsets of X with the finite intersection property has the property that $\cap_{A \in \mathcal{A}} A \neq \emptyset$*

PROOF. Since we only need the direction: "finite intersection property for collection of closeds ⇒ compact" we prove this and leave the other direction to the reader. Assume for contradiction that there exists an open cover $\mathcal{U} = \{U_i\}$ of X with no finite subcover. Consider the collection of closed subsets $\mathcal{A} := \{C_i := X \setminus U_i\}$. We claim that $\mathcal{A}$ has the finite intersection property, yet has empty intersection. The latter is clear from the fact that $\bigcup U_i = X$. But because $\mathcal{U}$ has no finite subcover, we must have that $\bigcap_{\textbf{finite}} C_i \neq \emptyset$. The other direction is [**Mun00**, Theorem 26.9]. □

Lemma 1.3.5 is an auxilliary notion of compactness which we will use in order to prove Proposition 1.3.3. What it says though, is that if we have a nested sequence of closed subsets $\cdots \subset \mathrm{C}_2 \subset \mathrm{C}_1$ then $\cap_{i=1}^{\infty} \mathrm{C}_n$ is nonempty whenever X is compact.

Proof of Proposition 1.3.3. Assume that X is compact and let F be a ultrafilter which is not convergent. This means that for each $x \in \mathrm{X}$ we can find an open set $\mathrm{U}_x$, containing $x$, such that $\mathrm{U}_x \notin \mathrm{F}$. Then $\{\mathrm{U}_x : x \in \mathrm{X}\}$ is an open cover of X which means that $\mathrm{X} = \bigcup_{i=1}^{n} \mathrm{U}_{x_i}$ by compactness. But Lemma 1.3.4 tells us that some $\mathrm{U}_{x_i}$ must be in F.

Now, assume that every ultrafilter converges. Let $\mathcal{A}$ be a collection of closed subsets of X with the finite intersection property, we claim that it has a nonempty intersection. This suffices to prove compactness by Lemma 1.3.5.

To see, this we claim that $\mathcal{A}$ is contained in an ultrafilter. Indeed, $\mathcal{A}$ is contained in a filter by first adding in arbitrary finite intersections, using the finite intersection property to avoid adding the empty set. We then add all the supersets and note that we have created a filter. Then Lemma 1.2.7 says that $\mathcal{A}$ is contained in an ultrafilter F. Now, $\mathrm{F} \to x$. We claim that $x \in \bigcap_{\mathrm{A} \in \mathcal{A}} \mathrm{A}$. For any $\mathrm{A} \in \mathcal{A}$, to prove that $x \in \mathrm{A}$ we need to show that for each open set U containing $x$, we have that $\mathrm{U} \cap \mathrm{A} \neq \emptyset$. We have that $\mathrm{U} \in \mathrm{F}$ because $\mathrm{F} \to x$. But $\mathrm{A} \in \mathcal{A} \subset \mathrm{F}$ and so $\mathrm{U} \cap \mathrm{A} \neq \emptyset$ because both are elements of the filter F. We are done. □

To prove Tychonoff's theorem we need the following important example of a filter.

Example 1.3.6 (Pushforward of filters). Let $f : \mathrm{X} \to \mathrm{Y}$ and F a filter on X. Consider

$$f_*\mathrm{F} := \{\mathrm{A} \subset \mathrm{Y} : f^{-1}\mathrm{A} \in \mathrm{F}\}$$

We call this the **pushforward filter**. In the next lemma, we show that it is indeed a filter.

Lemma 1.3.7. *Let $f : \mathrm{X} \to \mathrm{Y}$ be a continuous map of spaces and F a filter on X. Then $f_*\mathrm{F}$ is a filter and if F is an ultrafilter, $f_*\mathrm{F}$ is an ultrafilter.*

Proof. We verify the axioms of a filter and ultrafilter.

(1) The inverse image of $\emptyset$ is $\emptyset$ and hence $\emptyset$ cannot be in $f_*\mathrm{F}$.
(2) If $\mathrm{A} \in f_*\mathrm{F}$ and assume that $\mathrm{A} \subset \mathrm{B}$. Then $f^{-1}\mathrm{A} \subset f^{-1}\mathrm{B}$. Since $f^{-1}\mathrm{A} \in \mathrm{F}$ we must have that $f^{-1}\mathrm{B} \in \mathrm{F}$ and thus B is in $f_*\mathrm{F}$.
(3) Let $\mathrm{A}, \mathrm{B} \in f_*\mathrm{F}$, then $f^{-1}\mathrm{A}, f^{-1}\mathrm{B} \in \mathrm{F}$. Since F is a filter, $f^{-1}\mathrm{A} \cap f^{-1}\mathrm{B} \in \mathrm{F}$ which means that $\mathrm{A} \cap \mathrm{B} \in f_*\mathrm{F}$ since $f^{-1}(\mathrm{A} \cap \mathrm{B}) = f^{-1}(\mathrm{A}) \cap f^{-1}\mathrm{B}$.
(4) Assume that F is an ultrafilter; we shall use Proposition 1.2.5 to verify that $f_*\mathrm{F}$ is an ultrafilter. Let $\mathrm{A} \subset \mathrm{Y}$; if $\mathrm{A} \in f_*\mathrm{F}$ we are done and so we may assume that $\mathrm{A} \notin f_*\mathrm{F}$. We need to prove that $\mathrm{Y} \setminus \mathrm{A} \in f_*\mathrm{F}$. But now, $f^{-1}(\mathrm{Y} \setminus \mathrm{A}) = \mathrm{X} \setminus f^{-1}\mathrm{A}$ which must be in F because we had assumed that $\mathrm{A} \notin f_*\mathrm{F}$ and F is an ultrafilter.

□

Lemma 1.3.8. *Let $f : \mathrm{X} \to \mathrm{Y}$ be function between topological spaces. Then $f$ is continuous if and only if for any filter F on X such that $\mathrm{F} \to x$, we have that $f_*\mathrm{F} \to f_*x$.*

Proof. Assume that $f$ is continuous and assume that $\mathrm{F} \to x$. It suffices to prove that for any U an open subset of Y containing $f(x)$ that $\mathrm{U} \in f_*\mathrm{F}$. But now, $f^{-1}\mathrm{U}$ is an open subset containing $x$ and $\mathrm{F} \to x$ and therefore $f^{-1}\mathrm{U} \in \mathrm{F}$ as desired.

On the other hand, let $\mathrm{U} \subset \mathrm{Y}$ be an open subset. We want to prove that $f^{-1}\mathrm{U} \subset \mathrm{X}$ is open. Let $x \in f^{-1}\mathrm{U}$ and consider the principal ultrafilter $\mathrm{F}_x$. By definition $\mathrm{F}_x \to x$ and thus, by assumption $f_*\mathrm{F}_x \to f(x)$. The latter means that the principal ultrafilter corresponding to $f(x)$ is contained in $f_*\mathrm{F}_x$. Hence, for any open subset V containing $f(x)$, we have that $f^{-1}\mathrm{V} \in \mathrm{F}_x$. In particular, choose any open subset containing $f(x)$ and contained in U, then its inverse image is in $\mathrm{F}_x$ proving continuity of $f$. □

## 2. Tychonoff's theorem

We now present the long awaited proof of Tychonoff's theorem.

**2.1. Proof.** Let A be a nonempty set and $\{X_\alpha\}_{\alpha\in A}$ a collection of compact spaces. Our goal is to prove that $X := \prod_{\alpha\in A} X_\alpha$ is compact. Let $\mathcal{F}$ be an ultrafilter on X; by Proposition 1.3.3 we have to prove that it converges. We use the following criterion:

LEMMA 2.1.1. *As in the statement of Tychonoff's theorem. Let* F *be a filter on* X *and* $x \in X$. *Then* $F \to x$ *if and only if for all* $\alpha \in A$, *we have that* $\pi_{\alpha*}F \to \pi_\alpha(x)$.

PROOF. Assume that $F \to x$. Since the projection maps $\pi_\alpha : \prod X_\alpha \to X_\alpha$ are continuous, the claim follows from Lemma 1.3.8.

Assume that $\pi_{\alpha*}F \to \pi_\alpha(x)$ for all $\alpha$. We claim that $F \to x$. It suffices to prove that for any basic open of the product topology, containing $x$, is in F. But basic opens are of the form $\pi_{\alpha_1}^{-1}(V_1) \cap \cdots \cap \pi_{\alpha_j}^{-1}(V_1)$. Since $\pi_{(\alpha_i)*}F \to \pi_{\alpha_i}(x)$ by assumption, we conclude that each term in the intersection is in F. But F is closed under finite intersections and hence we are done. □

Now to complete the proof of Tychnoff's theorem, consider $\pi_{\alpha*}\mathcal{F}$. It is an ultrafilter on $X_\alpha$ by Example 1.3.6. By Proposition 1.3.3 it converges to some point $x_\alpha \in X_\alpha$. This determines a point $x \in X$ by universal properties of products. By assumption, $\pi_{\alpha*}\mathcal{F} \to \pi_\alpha(x) = x_\alpha$ and therefore, by Lemma 2.1.1, we have that $\mathcal{F} \to x$ proving the compactness of X by another application of Proposition 1.3.3.

**2.2. Cantor set 2: resolution of the interval.** Let C be the Cantor set; let us recall that any element $x \in C$ can be written as a (convergent) sum

$$x = \sum_{n=1}^{\infty} x_n/3^n \qquad x_n \in \{0, 2\}.$$

We will make use of Tychonoff's theorem and the topology of C to say something about the interval.

COROLLARY 2.2.1. *The Cantor set is compact. In particular, it is isomorphic to* $\prod_{\mathbb{N}}\{0, 1\}$.

PROOF. By construction of the Cantor, we have can construct a continuous bijection

$$h : C \xrightarrow{\cong} \{0, 1\}^{\mathbb{N}} = \prod_{\mathbb{N}}\{0, 1\}.$$

Indeed, it is given by

$$h(x = \sum_{n=1}^{\infty} 2y_n/3^n) = (x_n/2)_{n\in\mathbb{N}}$$

Its inverse is given by

$$h^{-1}((a_n)) = \sum_{n=1}^{\infty} 2a_n/3^n.$$

By the universal property of the product topology, $h$ is continuous and is easily seen to be a bijection. If we knew that the Cantor set was compact (we would know this if we had proved, using other means, that the interval is compact), then we would be done since any map from a compact to a Hausdorff which is a continuous bijection is a homeomorphism.

We leave this as homework. □

We now prove the compactness of the interval.

CONSTRUCTION 2.2.2 (The Cantor function). Let $x \in C$; then it is an element of $[0, 1]$ such that there exists a sequence $\{x_n\}$ for which $x_n \in \{0, 2\}$ and

$$x = \sum_{n=1}^{\infty} x_n/3^n.$$

We set

$$f_C : C \to [0, 1]$$

such that

$$f_{\mathrm{C}}(\sum_{n=1}^{\infty} x_n/3^n) = \sum_{n=1}^{\infty}(x_n/2)/2^n.$$

LEMMA 2.2.3. *The function $f_{\mathrm{C}}$ is continuous and surjective.*

PROOF. Let $y \in [0,1]$ and write it in binary:

$$y = \sum_{n=1}^{\infty} y_n/2^n \qquad y_n \in \{0,1\}.$$

Then setting

$$x = \sum_{n=1}^{\infty} 2y_n/3^n$$

we have that $f_{\mathrm{C}}(x) = y$. Continuity is an exercise. □

COROLLARY 2.2.4 (Compactness of the interval). *The interval $[0,1]$ is compact.*

PROOF. By Corollary 2.2.1, the Cantor set is compact, and by Lemma 2.2.3 the cantor function is a surjective continuous function onto $[0,1]$. Since the image of a compact set is compact, we have that $[0,1]$ must be compact.

□

The above Corollary is an example of a more general principle: for a compact Hausdorff space X, we will be able to to write down a nice surjective map

$$\{\text{Fractal space}\} \to \mathrm{X}.$$

This is a kind of a *resolution principle.* We have yet to define what fractal really means; it is what's called a **Stone space**. On first glance, it might seem like this is a ridiculous idea since we have some visual understanding of what compact Hausdorff spaces are, so why would we try to understand them via "weirder spaces" which are like the Cantor set? The answer is that: 1) these fractal spaces are basically describable using algebra and 2) this description is particularly useful when we want to expand the scope of nice spaces beyond compact Hausdorff spaces to **compactly generated weakly Hausdorff spaces**. The latter is the largest, most reasonable collection of spaces in which one can do topology.

**2.3. Stone-Čech compactification.** Before we make the above principle precise, we should see it in action. The following definition is very natural:

DEFINITION 2.3.1 (Compactification). Let X be a space. A **Stone-Čech compactification** (also called a **universal compactification**) is a compact Hausdorff space Y, equipped with a map $\iota : \mathrm{X} \to \mathrm{Y}$ such that for any other compact Hausdorff space C and a continuous map $f : \mathrm{X} \to \mathrm{C}$ there exists a unique map filling in the following commutative diagram

$$\begin{array}{ccc} \mathrm{X} & \xrightarrow{\iota} & \mathrm{Y} \\ \downarrow f & \swarrow \tilde{f} & \\ \mathrm{C} & & \end{array}.$$

Indeed, since we know that compact Hausdorff spaces are very nice, it is natural to look for the "closest compact Hausdorff" approximation to X. We now assume that X *is discrete* so, practically, X is just a set. Remarkably, universal compactifications can be reduced to this case, at least in most cases.

CONSTRUCTION 2.3.2 (Topology on ultrafilters). Let X be a set. Then we write $\beta(\mathrm{X})$ for the set of ultrafilters on X. We topologize it in the following way: the basic opens are sets of the form

$$[\mathrm{A}] := \{\mathrm{F} \in \beta(\mathrm{X}) : \mathrm{A} \in \mathrm{F}\}.$$

These indeed form a basic open thanks for to the following lemma.

LEMMA 2.3.3. *The following holds:*
*(1)* $[\emptyset] = \emptyset$*;*
*(2) If* $A \subset B \subset X$ *then* $[A] \subset [B]$*;*
*(3)* $[A \cup B] = [A] \cup [B]$*;*
*(4)* $[A \cap B] = [A] \cap [B]$
*(5)* $[X \smallsetminus A] = \beta(X) \smallsetminus [A]$.

This will be in the homework. We have a map $\iota : X \to \beta(X)$ given by $x \mapsto F_x$. Note that this map is continuous since X has the discrete topology. But there is more: we have an equality of sets $[\{x\}] = \{F_x\}$ and so $X \subset \beta(X)$ is a subspace inclusion! We propose $\beta(X)$ as a candidate for the universal compactification. In anticipation of this, we note the following result.

PROPOSITION 2.3.4 (Ultrafilter characterization of Hausdorfness). *Let* X *be a space, then* X *is Hausdorff if and only if* F *converges to* $x \in X$ uniquely*, if it exists.*

Recall that we had characterized compact spaces as those spaces where every ultrafilter admits a limit. Hausdorffness is characterized by saying that a limit must be unique, though no existence result is asserted.

PROOF. If X is Hausdorff, the fact that ultrafilters converges uniquely is the same proof as the case of sequences and will be left to the reader.

Now, assume that X is not Hausdorff, we construct an ultrafilter that converges to different points. Let $x, y \in X$ such that for any U, V opens with $x \in U$ and $y \in V$, $U \cap V = \emptyset$. Consider $F_{x,y} = \{U \subset X : U \text{ is open and either } x \in U \text{ or } y \in U\}$. Then $F_{x,y}$ is easily seen to be a filter and thus contained in an ultrafilter $F'_{x,y}$. But $F_x, F_y \subset F'_{x,y}$ and it converges to both $x$ and $y$. □

DEFINITION 2.3.5 (Ultralimit). Let X be compact, Hausdoff space and F an ultrafilter. Then the **ultralimit** of F, denoted by $\lim F \in X$ is the unique point (which exists!) such that $F \to \lim F$.

Therefore, given a map $f : X \to C$, the map $\tilde{f} : \beta(X) \to C$ is simply given by
$$F \mapsto \lim f_*(F).$$
This is an extremely simple idea.

THEOREM 2.3.6 (The Stone-Čech compactification of a discrete space). *Let* X *be a discrete space. The space* $\beta(X)$ *and the inclusion* $\iota : X \to \beta(X)$ *witnesses a universal compactification of* X.

PROOF. We prove this in turn:

(Step 1) We claim that $\beta(X)$ is compact. There are simpler ways to prove this, but we want to motivate the topology on $\beta(X)$ too which seems random. Let X be a set, let $I := \mathcal{P}(X)$. Then $\beta(X) \subset \mathcal{P}(I)$. Note that:
$$\mathcal{P}(I) = \prod_I \{0, 1\} = \{0, 1\}^I.$$
This is, in fact, compact by Tychonoff's theorem if we endow $\mathcal{P}(I)$ with the product topology. Now we claim that $\beta(X) \subset \mathcal{P}(I)$ is a subspace. To wards this end, we note that the basic opens of $\mathcal{P}(I)$ are given by specifying $\alpha_1, \cdots \alpha_n \in I$ (finite number of elements in I) and saying that for each $\alpha_j$ we specify $\{i_j\}$ where $i_j = 0$ or 1. We can then rewrite this as
$$[\alpha_1, \cdots, \alpha_\ell; \beta_1, \cdots, \beta_m] := \{(x_\alpha) : x_{\alpha_1} = x_{\alpha_2} = \cdots = x_{\alpha_\ell} = 1, x_{\beta_1} = \cdots x_{\beta_m} = 0\}.$$
We claim that each basic open $\beta(X)$ is the intersection of a basic open of $\mathcal{P}(I)$ with $\beta(X)$. Indeed, in our situation, for $A_1, \cdots, A_\ell, B_1, \cdots, B_m$ subsets of X we have that
$$[A_1, \cdots, A_\ell; B_1, \cdots B_m] = [F \in \mathcal{P}(I) : A_1, \cdots, A_\ell \in F, B_1, \cdots, B_m \notin F].$$

Hence, if F is an ultrafilter, we see that the following are all equivalent:
(a) $A_1, \cdots, A_\ell \in F, B_1, \cdots, B_m \notin F$
(b) $(X \smallsetminus A_1) \cap (X \smallsetminus A_2) \cap \cdots \cap (X \smallsetminus A_\ell) \cap B_1 \cap \cdots \cap B_n \in F.$
so that
$$[A] = [A_1, \cdots, A_\ell; B_1, \cdots B_m] \cap \beta(X).$$
So we need only prove that $\beta(X) \subset \mathcal{P}(I)$ is a closed subspace. If $Q \in \mathcal{P}(I) \smallsetminus \beta(X)$. Then Q is collection of subsets of X which is not an ultrafilter. So there must be some $A \subset B \subset X$ such that $A \in Q$ but $B \notin Q$. Then the open set $[A; B]$ contains Q but does not meet $\beta(X)$.

(Step 2) We claim that $\beta(X)$ is Hausdoff. If F, F′ are two ultrafilters which are not the same then there must be $A \subset X$ with $A \in F$ but $A \notin F'$. Then $F \in [A]$ but $F' \notin [A']$.

(Step 3) This is an extra step but very useful to know. We claim that $\iota(X) \subset \beta(X)$ is dense. This is equivalent to $\overline{\{\text{principal ultrafilters}\}} = \beta(X)$. To this end, say $F \in \beta(X)$. Then $F \in [A]$ so that $A \in F$ and $A \neq \emptyset$. So for any $x \in X$ we must have that $F_x \in [A]$. This proves the claim.

(Step 4) Given a map $f : X \to C$ where C is compact Hausdorff. We now construct $\tilde{f}$ as
$$\tilde{f} : \beta(X) \to C \qquad F \mapsto \lim f_*(F)$$
noting that $f_*(F)$ is an utlrafilter so this map is well-defined. Then the desired diagram commutes because $f(x) = \lim f_*(F_x)$ by unwinding defintions.

It remains to prove continuity. Since C is compact Hausdorff, it is normal and hence regular. So, for $x \in U$ where U is open, we can find an open V such that $x \in V \subset U$ and $\overline{V} \subset U$. We will use this observation. let $\tilde{f}(F) \in U$ where U is open. Find an open V containing $\tilde{f}(F)$ with the above property.

By definition, there exists an $A \in F$ such that $f(A) \subset V$. For any $F' \in [A]$ (so that $A \in F'$), we have that $\lim f_*(F') \subset \overline{V}$ which means that $\widetilde{f}([A]) \subset U$ and thus $\widetilde{f}$ is continuous at F.

□

Now, we want to find general universal compactifications of a space, not necessarily discrete. Let us consider $\beta(X)$ as a candidate; right off the bat there is something kind of fishy here: $\beta(X)$ has forgotten about the topology on X altogether. In fact, it is not clear at all if the map $X \to \beta(X)$ is continuous. Nonetheless, given $f : X \to C$ where C is compact Hausdorff, we can still define try to define a continuous map
$$\widetilde{f} : \beta(X) \to C$$
given by $F \mapsto \lim f_*(F)$. The key point to ensure that the map $\beta : X \to \beta(X)$ is continuous is just to declare that if to the eyes of all compact Hausdorff spaces C and all maps $f : X \to C$ that two ultrafilters F and G are the same. Then we declare them to be the same.

Construction 2.3.7 (Hausdorff quotient). Let X be a space. For any map $f : X \to C$ to a compact Hausdorff space C, we declare $\widetilde{f}$ to be the map
$$\beta(X) \to C \qquad F \mapsto \lim f_*(F).$$
On $\beta(X)$ let us declare the equivalence relation $F \sim Q$ if for all X and all compact Hausdorff space C we have that
$$\widetilde{f}(F) = \widetilde{f}(Q).$$
Endow $\overline{X} := \beta(X)/\sim$ with the quotient topology.

Theorem 2.3.8 (Existence of arbitrary universal compactifications). *Let* X *be a space. Then:*

*(1) the map* $\iota : X \to \beta(X) \to \overline{X}$ *is continuous;*
*(2) it witnesses* $\overline{X}$ *as a universal compactification of* X.

Proof. We claim: under the equivalence relation of Construction 2.3.7 if $\mathrm{F} \to x$, then $\mathrm{F} \sim \mathrm{F}'_x$. This is the whole point of the equivalence relation: indeed if $f : \mathrm{X} \to \mathrm{C}$ is any continuous map to a compact Hausdorff space then Lemma 1.3.8 says that $f_*(\mathrm{F}) \to f(x)$ and $f_*(\mathcal{F}'_x) \to f(x)$ and therefore $f_*(\mathrm{F})$ and $f_*(\mathcal{F}'_x)$ admits the unique ultralimit $f(x)$.

Let us see that this proves continuity of $\iota$. We use the converse of Lemma 1.3.8. Write

$$\mathrm{X} \xrightarrow{\iota'} \beta(\mathrm{X}) \xrightarrow{q} \overline{\mathrm{X}}.$$

Let $\mathrm{F} \to x$ where F is an ultrafilter on X. Then, unwinding definitions, we $\iota'_*(\mathrm{F}) \to \mathrm{F}$. But since $q$ is continuous by construction of the quotient topology we have that $q_*\iota'_*(\mathrm{F}) \to q(\mathrm{F})$. But now, $\mathrm{F} \sim \mathrm{F}'_x$ by the above result and thus $\iota_*(\mathrm{F}) = q_*\iota'_*(\mathrm{F}) \to \iota(\mathrm{F}) = \iota(x)$.

The universal property of $\overline{\mathrm{X}}$ follows by construction. Since the image of a compact set is compact, $\beta(\mathrm{X})$ is compact. It suffices to prove that $\overline{\mathrm{X}}$ is Hausdorff. By the ultrafilter characterization of Hausdorffness, it suffices to prove that any ultrafilter in $\overline{\mathrm{X}}$ has a unique limit. Now, suppose that $\mathcal{F} \in \beta(\beta(\mathrm{X})/\sim)$ is an ultrafilter with distinct limits $\overline{\mathrm{F}}, \overline{\mathrm{F}'}$ where they are, respectively, the equivalence class of $\mathrm{F}, \mathrm{F}'$ which are ultrafilters on X. Then there must exist some map $f : \mathrm{X} \to \mathrm{C}$ where C is compact Hausdorff such that $\lim f_*(\mathrm{F}) \neq \lim f_*(\mathrm{F}')$. But then, we have seen that the map $\widetilde{f} : \overline{\mathrm{X}} \to \mathrm{C}$ is continuous and thus $\widetilde{f}_*(\mathcal{F})$ has two limits points. This is not possible since C is Hausdorff. □

**2.4. Stone spaces; a glimpse into condensed mathematics.** We will now study a modern viewpoint on topological spaces; it comes under the moniker "condensed mathematics." Let X now be a compact Hausdorff space. Then one can construct the following: the identity map $\mathrm{X}^\delta \to \mathrm{X}$ is always continuous. Hence there exists, by Stone-Čech a continuous map

$$q : \beta(\mathrm{X}) \to \mathrm{X},$$

such that the following diagram of spaces and continuous functions commutes:

$$\begin{array}{ccc} \mathrm{X}^\delta & \xrightarrow{\mathrm{id}} & \mathrm{X} \\ \downarrow & \nearrow_{q} & \\ \beta(\mathrm{X}). & & \end{array}$$

By the commutativity of the above diagram, the map is surjective. Surjective maps of topological spaces should be thought of as a quotient: indeed, tautologically, the topological space of X is topology of the quotient of $\beta(\mathrm{X})$ by an equivalence relation: two ultrafilters F, G are equivalent if they have the same limit. Hence, one can attempt to describe all compact Hausdorff spaces by understanding them as quotients of spaces of the form $\beta(\mathrm{X})$.

There is a classical description of such spaces.

Definition 2.4.1. A **Stone space** is a compact, Hausdorff and totally disconnected space; this last part means that the only connected sets are singletons.

Example 2.4.2 (Cantor set). One of the first things that you showed in this class is that the Cantor set is, in fact, totally disconnected. Since we have seen a few times already that it is compact and Hausdorff, the cantor set is a Stone space.

Example 2.4.3 (The $p$-adic numbers). We discuss a very important space in mathematics. Let $p$ be a prime and $n$ a nonzero integer. We define the **$p$-adic valuation** of $n$, denoted by $\nu_p(n)$ to be the unique positive integer such that

$$n = p^{\nu_p(n)} n' \qquad \gcd(p, n') = 1.$$

While this looks intimidating, the intuition is quite simple: every integer has a unique prime factorization but, in some situations, we want to study one prime at a time. So $\nu_p(n)$ measures how $p$-divisible a number is. Hence having a large $p$-adic valuation means that one can "peel off" as much $p$ as possible from $n$.

The $p$-adic valuation of a rational number is then

$$\nu_p(m/n) = \nu_p(m) - \nu_p(n).$$

One also sets $\nu_p(0) = +\infty$. Hence the $p$-adic valuation is a certain function

$$\mathbb{Q} \to \mathbb{Z} \bigcup \{+\infty\}$$

which measures how $p$-divisible rational numbers are.

We want to turn a valuation into something topological. To do so, we introduce a device which, for all intends and purposes, capture the same information. The **$p$-adic absoute value** on $\mathbb{Q}$ is defined for $x \in \mathbb{Q}$

$$|x|_p := p^{-\nu_p(x)}.$$

The $p$-adic absolute value is an example of a **nonarchimedean absolute value**. It is a function

$$| - |_p : \mathbb{Q} \to \mathbb{R}_{\geqslant 0}$$

such that

(1) $|x|_p = 0$ if and only if $x = 0$;
(2) $|xy|_p = |x|_p|y|_p$
(3) $|x + y|_p \leqslant \max|x|_p, |y|_p$.

Since the $p$-adic valuation satisifies:

(1) $\nu_p(xy) = \nu_p(x) + \nu_p(y)$;
(2) $\nu_p(x + y) \geqslant \min\{\nu_p(x), \nu_p(y)\}$

we easily see that $| - |_p$ is a nonarchimedean absolute value. Part (3) of the definition of a nonarchimedean absolute value is called the **ultrametric triangle inequality** and is a refinement of the usual triangle inequality. One easy sees that $|x + y|_p \leqslant |x|_p + |y|_p$ also holds.

The absolute value on $\mathbb{Q}$ gives rise to a distance function $d_p(x, y) := |x - y|_p$ as in the usual situation. Just as one does in the real case, one can complete $\mathbb{Q}$ with respect to this metric.

Corollary 2.4.4. *There exists a field $\mathbb{Q}_p$ with a nonarchimedean absolute value $| - |_p$ which is complete as a metric space and $\mathbb{Q} \subset \mathbb{Q}_p$ is a dense subspace such that $| - |_p$ on $\mathbb{Q}_p$ extends $| - |_p$ on $\mathbb{Q}$.*

The miracle about $\mathbb{Q}_p$ is that it is totally disconnected and Hausdorff! The latter is not surprising since $\mathbb{Q}_p$ is a metric space. Let us examine total disconnectedness. First, we claim that if $|x + y|_p = \max|x|_p, |y|_p$ if $|x|_p \neq |y|_p$. Indeed, without loss of generality say that $|x|_p < |y|_p$. Suppose for contradiction that $|x + y|_p < |y|_p$. Then $|x|_p, |x + y|_p < |y|_p$. Therefore,

$$|y|_p = |y + x - x|_p \leqslant \max(|y + x|_p, -|x|_p) < |y|_p,$$

which is a contradiction.

This leads to two results:

Lemma 2.4.5. *On $\mathbb{Q}_p$ (or any field with a nonarchimedean absolute value) we have that:*

*(1) for any $w \in \mathrm{B}_r(v) = \{x : |x - v|_p < r\}$ we have that $\mathrm{B}_r(v) = \mathrm{B}_r(w)$.*
*(2) $\overline{\mathrm{B}}_r(v)$ is open in $\mathbb{Q}_p$.*

Proof. If $w \in \mathrm{B}_r(v)$ then for any $x \in \mathrm{B}_r(w)$ we have that

$$|x - v|_p = |x - w + w = v|_p \leqslant \max(|x - w|_p, |v - w|_p) < r.$$

This means that $\mathrm{B}_r(w) \subset \mathrm{B}_r(v)$. Running the argument the other way, we get the other inclusion.

For the second claim, we take any $x_0 \in \overline{\mathrm{B}}_r(v)$ and claim that $\mathrm{B}_r(x_0) \subset \overline{\mathrm{B}}_r(v)$. This will prove openness of the closed ball. To do so, since $x_0 \in \overline{\mathrm{B}}_r(v)$ we have that $|x_0 - v|_p \leqslant r$. Now if $|x - x_0| < r$ then

$$|x - v|_p = |x - x_0 + x_0 - v|_p \leqslant \max(|x - x_0|_p, |x_0 - v|_p) \leqslant r.$$

Which proves the claim. □

PROPOSITION 2.4.6. *The only connected set in $\mathbb{Q}_p$ are the singletons. In other words, it is totally disconnected.*

PROOF. Let $X \subset \mathbb{Q}_p$ be a subspace and assume that $v, w$ are two distinct points in X. Let $r$ be a real number such that $r < |v-w|_p$ which must exists since $0 < |v-w|_p$. This means that $X = (X \cap \{x : |x-v|_p \leqslant r\}) \cap (X \cap \{x : |x-v|_p > r\})$. By Lemma 2.4.5, $(X \cap \{x : |x-v|_p \leqslant r\})$ is an open subspace and both are nonempty (one contains $v$ the other $w$). Hence they constitute a separation of X. □

This is all pretty wild. The next example will make things more concrete.

EXAMPLE 2.4.7 (What are $p$-adic numbers, really?). As one suspects from the beginning a $p$-adic rational is simply a formal power series or, more informally, a thing that looks like

$$x = \sum_{i=k}^{\infty} a_k p^k \qquad k \in \mathbb{Z}, a_k \in \{0, \cdots, p-1\}.$$

Tracing through definitions, one sees that $|x| = p^{-k}$. In fact, one can prove that this expansion is unique [**?**, Corollary 4.3.4]. In this notation, the closed unit ball of radius $p^{-r}$ is given by

$$\overline{B}_{p^{-r}}(0) = \{\sum_{k=r}^{\infty} a_k p^k\}$$

We can use this expression to prove that $B_{p^{-r}}(0)$ is, in fact, compact. This leads to

THEOREM 2.4.8. *The p-adic unit disk*

$$\mathbb{Z}_p := \{x : |x|_p \leqslant 1\} = \overline{B}_1(0).$$

*is a Stone space.*

PROOF. Since the property of being totally disconnected (check this!) and Hausdorff is inherited by subspaces, it suffices to prove that $\mathbb{Z}_p$ is compact. One way to proceed is to prove that $\mathbb{Z}_p$ is a closed (easy) and totally bounded subset of $\mathbb{Q}_p$ and show that this implies compactness in a metric space. We work differently. Let $\mathbb{F}_p^{\times} = \{0, 1, \cdots, p-1\}$. Then we claim that there is a homeomorphism

$$\{\sum_{k=r}^{\infty} a_k p^k\} \xrightarrow{\cong} \prod_{\mathbb{N}} \mathbb{F}_p^{\times}.$$

We then obtain the result via Tychnoff's theorem. We refer to `https://math.stackexchange.com/questions/78449/why-are-closed-balls-in-the-p-adic-topology-compact` for a full proof. □

We return to our discussion of Stone-Čech compactifications.

DEFINITION 2.4.9 (Gleason). A space is **extremally disconnected** if every open set $U \subset X$ has an open closure.

LEMMA 2.4.10. *Let X be a compact Hausdorff space. The following are equivalent:*

*(1) X is extremally disconnected;*
*(2) any surjection $f : X' \to X$ from a compact Hausdorff space $X'$ admits a splitting, i.e., there exists a continuous map $s : X \to X'$ such that $f \circ s = \mathrm{id}_X$.*

*Furthermore, any extremally disconnected space which is Hausdorff must be totally disconnected.*

Any compact Hausdorff space satisfying property (2) is called **projective**.

PROOF. We first prove the "furthermore." Suppose that $A \subset X$ is a subspace containing at least two elements. Since X is Hausdorff, A is as well. Let $x, y$ be these distinct elements so that, by the Hausdorff assumption, we can find $U_x, U_y$ open neighborhoods of $x$ and $y$ in X that separates them. Here's the key property of extremally disconnected spaces that helps us: given any two disjoint opens of X, their closures are disjoint. So $\overline{U}_x$ and $\overline{U}_y$ are disjoint closed containing $x$ and $y$ respectively. But, by assumption of extremally disconnected, we see that they are also open. Hence $\overline{\mathrm{U}}_x \cap \mathrm{A}$ and $\overline{\mathrm{U}}_y \cap \mathrm{A}$ constitute a separation of A.

To prove the claim: let U, V be two disjoint opens of X. Then $\mathrm{V} \subset \mathrm{X} \smallsetminus \mathrm{U}$. Since $\mathrm{X} \smallsetminus \mathrm{U}$ is closed we have that $\overline{\mathrm{V}} \subset \mathrm{X} \smallsetminus \mathrm{U}$. But now, $\overline{\mathrm{V}}$ is open and thus $\mathrm{X} \smallsetminus \overline{\mathrm{V}}$ is closed. Since U is contained in this closed subset, we have that its closure is again contained in $\mathrm{X} \smallsetminus \overline{\mathrm{V}}$, proving the claim.

The proof of the main part of the Lemma is due to Gleason and can be found in [**?**, Tag 08YN]. Let us sketch a proof (2)⇒(1). Let X be projective and $\mathrm{U} \subset \mathrm{X}$ is open. The goal is to prove that $\overline{\mathrm{U}}$ is also open. Take $\mathrm{X} \times \{0,1\}$ which is again compact and Hausdorff. Consider

$$\mathrm{Y} := (\mathrm{X} \smallsetminus \mathrm{U}) \times \{0\} \cup \overline{\mathrm{U}} \times \{1\} \subset \mathrm{X} \times \{0,1\}.$$

Then we have a surjective, continuous map

$$\mathrm{Y} \subset \mathrm{X} \times \{0,1\} \to \mathrm{X}$$

where the second map is the projection. Indeed, the image is $(\mathrm{X} \smallsetminus \mathrm{U}) \cup \overline{\mathrm{U}}$. By projectivity, we get a splitting $s : \mathrm{X} \to \mathrm{Y}$. One then proves, by tracing diagrams that

$$s^{-1}(\overline{\mathrm{U}} \times \{1\}) = \overline{\mathrm{U}}.$$

But then $\overline{\mathrm{U}} \times \{1\} = \mathrm{Y} \cap (\mathrm{X} \times \{1\})$ and so is an open subspace of Y and we are done. □

The (2)⇒(1) direction helps us prove that $\beta(\mathrm{X})$ is extremally disconnected as given in the homework.

Example 2.4.11 (The space $\beta(\mathrm{X})$). In the homework, one is led to prove that if X is discrete, $\beta(\mathrm{X})$ is an extremally disconnected space which is furthermore compact and Hausdorff. In particular, it is totally disconnected.

2.4.12. *Condensed mathematics.* In 2018, Clausen and Scholze revolutionzed the way that we think about topological spaces and algebra by introducing condensed mathematics. The problem they seek to solve is how can one do algebra and topology at the same time. Let us sketch the definition of a **condensed set**:

Definition 2.4.13. A **condensed set** X is a rule that assigns to each extremally disconnected space T a set

$$\mathrm{T} \mapsto \mathrm{X}(\mathrm{T});$$

any continuous function $f : \mathrm{T} \to \mathrm{T}'$ a function (just of sets)

$$\mathrm{X}(\mathrm{T}') \xrightarrow{\mathrm{X}(f)} \mathrm{X}(\mathrm{T})$$

such that:

(1) $\mathrm{X}(\mathrm{id}) = \mathrm{id}$;
(2) $\mathrm{X}(f \circ g) = \mathrm{X}(g) \circ \mathrm{X}(f)$;
(3) $\mathrm{X}(\emptyset) = *$
(4) there are natural map "insert" maps $\iota_1 : \mathrm{T}_1 \to \mathrm{T}_1 \sqcup \mathrm{T}_2$ and $\iota_2 : \mathrm{T}_2 \to \mathrm{T}_1 \sqcup \mathrm{T}_2$, inducing
$$\mathrm{X}(\mathrm{T}_1 \sqcup \mathrm{T}_2) \xrightarrow{(\mathrm{X}(\iota_1), \mathrm{X}(\iota_2))} \mathrm{X}(\mathrm{T}_1) \times \mathrm{X}(\mathrm{T}_2).$$
We ask that tis map is an isomorphism.

This is not quite a definition because of set-theoretic issues but if one is willing to ignore it, then this is a definition of a condensed set. Condensed mathematics then builds structures upon condensed sets and "does mathematics" based on condensed sets rather than usual sets. In particular one can contemplate objects such as **condensed abelian groups**; in the Definition **??** above just ask that X(T) is an abelian group for all T and the functions are just homomorphisms of abelian groups. One of the key results of condensed mathematics is that the following object is a condensed set: take any compact Hausdorff space X then consider

$$\mathrm{T} \mapsto \{\text{continuous functions } \mathrm{T} \to \mathrm{X}\}.$$

Precomposition is used to define $\mathrm{X}(f)$. The miracle is that this assignment *recovers* X *back as a space* in the usual way. Hence, in this version of mathematics, compact Hausdorff spaces are included as your "bare-bones" sets.

APPENDIX A

# Some set theory

APPENDIX B

# Some category theory

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