Differentiable Manifolds

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Introduction

These are notes for MATD67H3, the UTSC version of a first class in differentiable manifolds.

CHAPTER 2

What is differential topology?

1. Motivation: exotic spheres

One of the most important and beautiful collection of spaces in mathematics are the spheres. We are all familiar with the circle

$$\mathrm{S}^1 := \{(x, y) : x^2 + y^2 = 1\} \subset \mathbb{R}^2.$$

This is an example of a **differentiable manifold**, the main object of study in this class. What makes the circle "nice" (or, really, what makes it a differentiable manifold) is that around each point $x \in \mathrm{S}^1$ there exists a neighborhood around x which looks like the interval $(a, b) \subset \mathbb{R}$. To make this more precise let us recall the trigonometric identity

$$\cos(t)^2 + \sin(t)^2 = 1.$$

Then, roughly speaking, we can define a function

$$\theta_+ : (0, 2\pi) \to \mathbb{R}^2 \qquad t \mapsto (\cos(t), \sin(t))$$

whose image *almost* hits all S^1. The main property of this map is that it is a bijection onto its image; even better it is a something called a **diffeomorphism**. This is a concept which we will introduce later on in class but it is a bijection that preserves "all possible differentiable information" between the two spaces. Now, note, that $(\cos(0), \sin(0)) = (\cos(2\pi), \sin(2\pi)) = (1, 0)$ and hence we are missing the point $(1, 0)$. The extension of θ_+ to $[0, 2\pi)$ is a bijective map, but its inverse is *not continuous*: if it were, one can show that its image had to be open which is not the case. Hence we cannot parametrize S^1 fully using any one open subset of $\mathbb{R}$.

To resolve this problem, consider another function

$$\theta_- : (0, 2\pi) \to \mathbb{R}^2 \qquad t \mapsto (\cos(t + \pi), \sin(t + \pi))$$

which does contain the point $(1, 0)$ in the image. The union of the images of θ_- and θ_+ turns out to cover all of S^1. This is a key difference in analysis versus differential topology: whereas analysis studies intervals $(a, b) \subset \mathbb{R}$ and functions on them, spaces that one encounters in differential topology will require "gluing" together these intervals. This is where the rubber meets the road: how intervals and their generalizations (open subsets $\mathrm{U} \subset \mathbb{R}^n$) are glued together is what makes differential topology both interesting and difficult.

To make all of this precise, we will introduce the notion of a **differentiable structure** on topological spaces. Roughly speaking a differentiable structure introduces to each point $x \in \mathrm{S}^1$ the concept of an *infinitely differentiable function* $f : \mathrm{S}^1 \to \mathbb{R}$ *around* x which are *consistent as* x *varies*. For example: if $x \neq (1, 0)$ then x is in the image of θ_+ and we can compose f with θ_+ to get

$$f \circ \theta_+ : (0, 2\pi) \to \mathbb{R}.$$

Now we open up a book in analysis and there is a notion (to be reviewed today) of what it means for this composite to be differentiable, continuously differentiable or even infinitely differentiable (smooth)! In this sense, θ_+ is used to "coordinatize" the function f and we are happy if the resulting coordinatization of f is smooth. To explain "consistency" will take us further afield, but it concerns how different coordinatization varies (roughly, the overlaps of the images of θ_- and θ_+).

For our informal discussion, we assert that anytime we have a set $\mathrm{Z} \subset \mathbb{R}^n$ defined by some polynomial equation (in the above case Z is defined by $x^2 + y^2 = 1$) then Z inherits a **standard differentiable structure**. Roughly: for any $x \in \mathrm{Z}$ and any open subset $\mathrm{U} \subset \mathrm{Z}$ (these are of the form $\mathrm{U} \cap \mathrm{Z}$ where $\mathrm{U} \subset \mathbb{R}^n$ is a union of open balls in $\mathbb{R}^n$), then we declare differentiable functions to be those that come from infinitely differentiable functions $f : \mathrm{U} \to \mathbb{R}$. Of fundamental interest are the spheres:

$$\mathrm{S}^n = \{(x_1, x_2, \cdots, x_{n+1}) : x_1^2 + x_2^2 + \cdots x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1},$$

with its standard differentiable structure.

Now, when $n = 1$, we have the standard differentiable structure, and also one defined in the previous paragraph using θ_+ and θ_-. It turns out that they are **diffeomorphic**. In topology, we have the notion of a **homeomorphism**: two spaces X and Y are homeomorphic if there are continuous functions

$$f : \mathrm{X} \to \mathrm{Y} \qquad g : \mathrm{Y} \to \mathrm{X}$$

such that $g \circ f, f \circ g$ are the identities. A diffeomorphism is a homeomorphism that preserves differentiable structure. For example, f, g could both be the identity map on underlying sets, but one can imagine equipping X and Y with *different* differentiable structures and X and Y would not be a diffeomorphic. In the case of the sphere, we say that a differentiable structure on S^n is **exotic** if it is not diffeomorphic to the standard structure.

Question 1.0.1. Is there any exotic differentiable structure on the sphere?

It turns out that when $n = 1, 2, 3, 5, 6$ (Note that 4 is skipped) one cannot find any exotic differentiable structure on S^n. Milnor shocked mathematics when he proved:

Theorem 1.0.2 (Milnor 1956, [**Mil56**]). *There exists an exotic differentiable structure on* S^7.

The construction is not easy and even proving that his sphere is not the standard one requires technology outside the scope of this class. Roughly, however, Milnor's exotic 7 sphere is constructed out of arranging S^3 and S^4 to "glued" in a strange way. He then proves (and in fact develops an entire technology for proving) that it is not the **boundary** of any 8-dimensional differentiable manifold: note that S^n is the boundary of

$$\mathrm{D}^{n+1} := \{(x_1, x_2, \cdots, x_{n+1}) : x_1^2 + x_2^2 + \cdots x_{n+1}^2 \leqslant 1\}.$$

The concept of a manifold with boundary is something we will encounter in this class. In fact, this result comes as a surprise to Milnor himself. At this point in history, we have understood smooth structures better than ever before. As a sampler: there are 27 exotic smooth structures on S^7, while there are 523263 on S^{19} and we know that S^{61} is the last odd-dimension where *no* exotic smooth structure can exist (we only knew this in 2017). But a question remains:

Question 1.0.3. Is there an exotic 4-sphere?

2. Some recollections on analysis

The first order of business is to define what a differentiable manifold is. We say the words and then fill in the meaning.

Definition 2.0.1. A space X is a **topological manifold of dimension** n if it is a second countable Hausdorff space such that for any $x \in \mathrm{X}$ there exists an open $\mathrm{V}_x \subset \mathrm{X}$ for which V_x is homeomorphic to an open subset U_x of $\mathbb{R}^n$. A **chart** on X is a triple $(\mathrm{U}, \mathrm{V}, \varphi)$ where $\mathrm{U} \subset \mathbb{R}^n$ is an open, $\mathrm{V} \subset \mathrm{X}$ is an open and $\varphi : \mathrm{U} \to \mathrm{V}$ is a homeomorphism. An **atlas** for X is a collection of charts $\{(\mathrm{U}_\alpha, \mathrm{V}_\alpha, \varphi_\alpha)\}$ such that $\bigcup_\alpha \mathrm{V}_\alpha = \mathrm{X}$. An atlas is said to be **smooth** if for any indices α, β the map

$$\psi_{\alpha\beta} := \varphi_\beta^{-1}\varphi_\alpha : \varphi_\alpha^{-1}(\mathrm{V}_\alpha \cap \mathrm{V}_\beta) \to \varphi_\beta^{-1}(\mathrm{V}_\alpha \cap \mathrm{V}_\beta)$$

is a smooth.

Definition 2.0.1 is a combination of both analytic and topological definitions. So we begin by reviewing some analysis and some topology. In particular, our first order of business is to explain the following concepts:

(1) a topological space;
(2) a second countable space;
(3) a Hausdorff space;
(4) smooth maps.

We will start from the last concept, which is analytic in nature. We will give references for the proofs of these statements but may not necessarily give them in full; we refer to [**Spi65**] for a textbook reference. A key theme in this class is that we are trying to *globalize analysis with multiple variables.*

2.1. Reminders on differential analysis in one variable. Let us recall the following basic definitions in one variable:

DEFINITION 2.1.1. A function $f : \mathbb{R} \to \mathbb{R}$ is **continuous at** x_0 if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for any

$$y \in (f(x_0) - \epsilon, f(x_0) + \epsilon),$$

there exists $x \in (x_0 - \delta, x_0 + \delta)$ such that $f(x) = y$.

We have seen that this is the key definition in analysis and arguably the main object of study. It says that for any ϵ-neighborhood of $f(x_0)$, there exists a δ-neighborhood around x_0 such that any y in the ϵ-neighborhood, it is realized as a value of f. This definition is *local* in nature: it makes no claim on whatever happens outside of a small neighborhood of x_0. In particular if $(a, b) \subset \mathbb{R}$ is an open interval we can ask that f is continuous on every point of (a, b). Though making "small neighborhood" a precise concept is not easy. In particular, even if f is only defined on a subset $\mathrm{A} \subset \mathbb{R}$ we know how to make sense of continuity: we ask that for any $x_0 \in \mathrm{A}$ and any ϵ-neighborhood of $f(x_0)$, there exists a $\delta > 0$ such that for any y in the ϵ-neighborhood, it can be realized as a a value of f on $\mathrm{A} \cap (x_0 - \delta, x_0 + \delta)$.

Continuous functions are often pathological — the functions that appear in daily life, or even mathematics, are of the following form

(1) trigonometric functions like $\sin, \cos, \tan$,
(2) exponential function $\exp$,
(3) polynomials: $f(x) = 27x^5 + 6x^2 + 7$ and so on.

These functions are **differentiable**, which is a kind of "niceness" or "regularity" condition that one can put on functions.

DEFINITION 2.1.2. A function $f : \mathbb{R} \to \mathbb{R}$ is **differentiable at** x_0 if the limit

$$\lim_{h \to 0} \tfrac{f(x_0+h)-f(x_0)}{h}$$

exists; in which case we denote its value by $f'(x_0)$.

Again, the notion of a differentiable function is local in nature: it only demands "good behavior" around x_0. We recall that if f is differentiable at x_0, then f must be continuous at x_0 although the converse is false. Now, suppose that $\mathrm{U} \subset \mathbb{R}$ is an open subset and f is a function $f : (a, b) \to \mathbb{R}$, then we can ask that f is differentiable everywhere on U, i.e., for any $x_0 \in (a, b)$, $f'(x_0)$ exists. Then we can ask that the function

$$\mathrm{A} \to \mathbb{R} \qquad x_0 \mapsto f'(x_0)$$

is continuous. In this case, we say that f is **continuously differentiable**. More generally, if $\mathrm{A} \subset \mathbb{R}$ is a subset, then f is continuously differentiable if for all $x \in \mathrm{A}$, there exists a small open interval (a, b) around x and an continuously differentiable map $\mathrm{F} : (a, b) \to \mathbb{R}$ such that $\mathrm{F}|_{(a,b)\cap \mathrm{A}} = f|_{(a,b)\cap \mathrm{A}}$. The notation for this is:

$$f \in \mathcal{C}^1(\mathrm{A})$$

and we read "f is a C^1-function." If we set $\mathcal{C}^0(A)$ to be the collection of continuous function, then f is C^1 if and only if f' is in $\mathcal{C}^0$. So we say that $f \in \mathcal{C}^k(A)$ if $f' \in \mathcal{C}^{k-1}(A)$. This means that f is k-times differentiable with all its derivatives continuous. The collection $\mathcal{C}^\infty(A)$ is then defined to be those functions on f which is in $\mathcal{C}^k(A)$ for all k. This means that all derivatives of f exists and are continuous; in this case we say that f is a **smooth** function. All the functions listed above are indeed smooth. We now generalize the definition of a smooth function to the multivariate context.

2.2. Reminders on differential analysis in multiple variables. Now, we want to do all of this with multiple variables. Presumably you have already done this in your analysis class and maybe found this intimidating. Let me remark that in this class we will only be performing "soft analysis" — this means that we will hardly ever find an explicit $\epsilon - \delta$ estimate and content ourselves with very very rough estimates (or no estimate at all).

EXAMPLE 2.2.1 (Soft analysis). Most arguments in your first analysis class will involve choosing suitable values/estimates for ϵ and δ. One of the things that this is good for is that it keep us honest and not make mistakes, an important task for any mathematician. In this class, we assume some familiarity with analysis and thus we allow ourselves to perform soft arguments. For example, recall that we have the intermediate value theorem:

THEOREM 2.2.2 (Intermediate value theorem). *Let $f : [a, b] \to \mathbb{R}$ be a continuous function, then for any x in between $f(a)$ and $f(b)$ there exists an element $c \in [a, b]$ such that $f(c) = x$.*

A soft analysis proof goes like this: since f is continuous and $[a, b]$ is connected, $f([a, b])$ is connected and thus any element in between the values $f(a)$ and $f(b)$ must be hit. Presumably the reader can fill in the $\epsilon - \delta$ details but we will not demand that in this class.

Now, a function $f : \mathbb{R}^n \to \mathbb{R}^m$ can be represented as

$$f(x_1, \cdots, x_n) = (f_1(x_1, \cdots, x_n), \cdots, f_m(x_1, \cdots, x_n)).$$

Spelling this out, this means that we are given an m-tuple of functions which has n-variables as input. At this point you should be somewhat familiar with the notation

$$B_\epsilon(x_0) := \{x \in \mathbb{R}^m : |x - x_0| < \epsilon\} \subset \mathbb{R}^m.$$

This is called the "ball of radius ϵ" and $| - |$ refers to the norm in m-space. This is the generalization of an ϵ-interval in one variable. The notion of a continuous function is easy enough to generalize:

DEFINITION 2.2.3. Let $U \subset \mathbb{R}^n$ be an open subset. A function $f : U \to \mathbb{R}^m$ is **continuous at** $x_0 \in U$ if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for any $y \in B_\epsilon(f(x_0))$, there exists $x \in B_\delta(x_0) \subset U$ such that $f(x) = y$. More generally, if A is any subset of $\mathbb{R}^n$, it is continous at $x_0 \in A$ if there exists an open subset U of $\mathbb{R}^n$ containing x_0 and a continuous function $F : U \to \mathbb{R}^n$ such that $F|_{U \cap A} = f|_{U \cap A}$.

This is all good, but in the interest of only performing soft analysis arguments we note the following formulation of continuity.

LEMMA 2.2.4. *Let $A \subset \mathbb{R}^n$. A function $f : A \to \mathbb{R}^m$ is continuous if and only if for any $U \subset \mathbb{R}^m$, there exists an open $V \subset \mathbb{R}^n$ such that $f^{-1}(U) = V \cap A$.*

PROOF. Here's the key point: a subset $W \subset \mathbb{R}^k$ is open if and only if it can be written as a union $\bigcup_{w \in W} B_\epsilon(w) = W$. □

Lemma 2.2.4 says that continuity is the preservation of a topological property of subsets, namely of openness. It is more abstract than the $\epsilon - \delta$ argument but also conceptually simpler. As an application, we give a soft-analysis argument of the following result which might be familiar in an analysis class.

Corollary 2.2.5 (Maximum value theorem). *Let* $\mathrm{A} \subset \mathbb{R}^n$ *be a compact subset and* $f : \mathrm{A} \to \mathbb{R}^m$ *continuous. Then* $f(\mathrm{A})$ *is compact. In particular, it contains the supremum.*

Proof. Let $\mathcal{O} = \{\mathrm{V}_\alpha\}_{\alpha\in\mathrm{A}}$ be an open cover of $f(\mathrm{A})$. Since f is continuous, for each open V_α, we have that $f^{-1}(\mathrm{V}_\alpha)$ is open. In particular $\{f^{-1}(\mathrm{V}_\alpha)\}$ is an open cover of A. But now, A is compact, hence it refines to a finite cover $f^{-1}(\mathrm{V}_j)$, $j = 1, \cdots, n$. Then $\{\mathrm{V}_j\}_{j=1,\cdots,n}$ is an subcover of $\mathcal{O}$ which covers $f(\mathrm{A})$. □

We will revisit these ideas in a more abstract context in the next lecture. But we now discuss differentiability. To motivate this definition we reformulate Definition 2.1.2. In the notation of that definition, let us consider the linear function:

$$\mathrm{D}f_{x_0} : \mathbb{R} \to \mathbb{R} \qquad \lambda(h) = f'(x_0) \cdot h.$$

This is a linear function that simply scales vectors by the quantity $f'(x_0)$. This might seem puzzling at first since $\mathrm{D}f_{x_0}$ is basically the same information as $f'(x_0)$; the latter is the value of the former at 1. It turns out that, for our purposes, the right way to view the derivative at a point is not as a number, in other words we don't really want to think of the derivative as a function $x \mapsto f'(x)$. But we want to think of as a function $x \mapsto \mathrm{D}f_x$ where the target is a linear function. Do not worry about this for now, but note that a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable if and only if

$$\lim_{h\to 0} \tfrac{f(x_0+h)-f(x_0)-\lambda(h)}{h} = 0.$$

Hence we can reformulate the notion of a function being differentiable at x_0 by saying that there exists a linear function $\mathrm{D}f_{x_0} : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{h\to 0} \tfrac{f(x_0+h)-f(x_0)-\mathrm{D}f_{x_0}(h)}{h} = 0.$$

One of the first results that one proves about this definition is that $\mathrm{D}f_{x_0}$ is, in fact, unique and so the notation is justified; we also note that differentiable functions are continuous.

In any case, the higher dimension generalization of being differentiable is as follows.

Definition 2.2.6. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **differentiable at** $x_0 \in \mathbb{R}^n$ if there exists a linear transformation

$$\mathrm{D}f_{x_0} : \mathbb{R}^n \to \mathbb{R}^m$$

such that

$$\lim_{h\to 0} \tfrac{|f(x_0+h)-f(x_0)-\mathrm{D}f_{x_0}(h)|}{|h|} = 0.$$

We call the transformation $\mathrm{D}f_{x_0}$ the **derivative of** f **at** x_0 if it exists. This linear transformation is unique.

Lemma 2.2.7. *The derivative is unique.*

Proof. Let $\mathrm{T} : \mathbb{R}^n \to \mathbb{R}^m$ be a transformation that satisfies

$$\lim_{h\to 0} \tfrac{|f(x_0+h)-f(x_0)-\mathrm{T}(h)|}{h} = 0.$$

Rewriting $d(h) := f(x_0 + h) - f(x_0)$ we have that

$$\lim_{h\to 0} \tfrac{|\mathrm{T}(h)-\mathrm{D}f_{x_0}(h)|}{|h|} = \lim_{h\to 0} \tfrac{|\mathrm{T}(h)-d(h)+d(h)-\mathrm{D}f_{x_0}(h)|}{|h|} \leqslant \lim_{h\to 0} \tfrac{|\mathrm{T}(h)-d(h)|}{|h|} + \lim_{h\to 0} \tfrac{|d(h)-\mathrm{D}f_{x_0}(h)|}{|h|} \leqslant 0.$$

Therefore, for any $x \neq 0$ we have, by linearity, that:

$$\lim_{h\to 0} \tfrac{|\mathrm{T}(hx)-\mathrm{D}f_{x_0}(hx)|}{|hx|} = \lim_{h\to 0} \tfrac{|h|}{|h|} \tfrac{|\mathrm{T}(x)-\mathrm{D}f_{x_0}(x)|}{|x|} = \tfrac{|\mathrm{T}(x)-\mathrm{D}f_{x_0}(x)|}{|x|},$$

and thus $\mathrm{T} = \mathrm{D}f_{x_0}$ as linear transformations. □

The derivative satisfy generalizations of usual properties that we have seen in one variable calculus. We refer to [**Spi65**, Theorem 2-2, 2-3] for statements and proofs of these.

2.3. The Jacobian and partial derivatives. Since $\mathrm{D}f_{x_0} : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation we can, up to choosing bases, represent $\mathrm{D}f_{x_0}$ as a matrix. This matrix, in fact, encodes important properties about the derivative. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a function, then we can hold all variables constant in the domain except at the i-th spot, i.e., fix $a_1, \cdots, a_{i-1}, a_{i+1}, \cdots a_n \in \mathbb{R}$ and consider the function

$$f : \mathbb{R} \to \mathbb{R} \qquad x \mapsto f(a_1, \cdots, a_{i-1}, x, a_{i+1} \cdots a_n).$$

In this way, the ***i*-th partial derivative** at $x_0 = (a_1, \cdots, a_n)$ is the limit

$$\lim_{h \to 0} \tfrac{f(a_1, \cdots, a_{i-1}, a_i + h, a_{i+1} \cdots a_n) - f(a_1, \cdots, a_{i-1}, a_i, a_{i+1} \cdots a_n)}{h} =: \mathrm{D}_i(f)(x_0)$$

Now, write $f : \mathbb{R}^n \to \mathbb{R}^m$ as $f = (f^1, \cdots, f^m)$ noting that a map into $\mathbb{R}^m$ is the same thing as m-functions $f^1, \cdots, f^m : \mathbb{R}^n \to \mathbb{R}$.

PROPOSITION 2.3.1. *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at x_0, then $\mathrm{D}_j f^i(x_0)$ exists and the Jacobian matrix is given by:*

$$\begin{pmatrix} \mathrm{D}_1 f^1(x_0) & \mathrm{D}_2 f^1(x_0) & \cdots & \mathrm{D}_n f^1(x_0) \\ \mathrm{D}_1 f^2(x_0) & \cdots & \cdots & \mathrm{D}_n f^2(x_0) \\ \cdots & \cdots & \cdots & \cdots \\ \mathrm{D}_1 f^m(x_0) & \cdots & \cdots & \mathrm{D}_n f^m(x_0). \end{pmatrix}$$

PROOF. Apply the chain rule; see [**Spi65**, Theorem 2-7]. □

REMARK 2.3.2. If $f : \mathbb{R} \to \mathbb{R}$, we have reformulated the derivative as a linear transformation $\mathrm{D}f_{x_0} : \mathbb{R} \to \mathbb{R}$. The Jacobian matrix corresponding to this linear transformation is exactly the 1×1-matrix given whose only entry is the number $f'(x_0)$. In this sense the Jacobian matrix is a generalization of the maneuver we made above to go from the number $f'(x_0)$ to the linear transformation $\mathrm{D}f_{x_0}$.

We remark that the converse to Proposition 2.3.1 fails: there exists a function whose partials exist but need not be differentiable. The following theorem guarantees existence as soon as we know that the partials are continuous.

THEOREM 2.3.3 (Differentiability theorem). *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a function, then $\mathrm{D}f_{x_0}$ exists if the limits $\mathrm{D}_j f^i(x_0)$ exists for all i, j in an open set containing x_0 (call it U) and the function $\mathrm{D}_j f^i : \mathrm{U} \to \mathbb{R}^n$ is continuous.*

PROOF. See [**Spi65**, Theorem 2-8]. □

DEFINITION 2.3.4. Let $\mathrm{U} \subset \mathbb{R}^m$ be an open subset. A function $f : \mathrm{U} \to \mathbb{R}^n$ is **continuously differentiable at** x_0 if the hypothesis of Theorem 2.3.3 holds. If it is continuously differentiable at all points of U, then we say that f is **continuously differentiable**. As usual if A is any subset and $f : \mathrm{A} \to \mathbb{R}^m$ is a function, we say that it is **continuously differentiable at** A if for all $x \in \mathrm{A}$ there exists an open $\mathrm{U} \subset \mathbb{R}^m$ and a continuously differentiable function $\mathrm{F} : \mathrm{U} \to \mathbb{R}^m$ such that $\mathrm{F}|_{\mathrm{U} \cap \mathrm{A}} = f|_{\mathrm{U} \cap \mathrm{A}}$. We write

$$f \in \mathcal{C}^1(\mathrm{A}).$$

In other words, if $\mathrm{A} \subset \mathbb{R}^m$ a function $f : \mathrm{A} \to \mathbb{R}^m$ is $\mathcal{C}^1(\mathrm{A})$ is all its partial derivatives exists and are continuous. To iterate this definition, we have to consider mixed partials: given $f : \mathrm{A} \to \mathbb{R}$, assume that $\mathrm{D}_i f : \mathrm{A} \to \mathbb{R}$ exists. Then we can ask if its derivative exists, this time maybe with respect to the j-th coordinate. If it exists we write:

$$\mathrm{D}_{j,i} f := \mathrm{D}_j(\mathrm{D}_i f) : \mathrm{A} \to \mathbb{R}.$$

LEMMA 2.3.5 (Symmetry of second derivatives). *Assume that $\mathrm{D}_{j,i} f$ and $\mathrm{D}_{i,j} f$ are continuous in an open set containing x_0. Then*

$$\mathrm{D}_{j,i} f(x_0) = \mathrm{D}_{i,j} f(x_0).$$

PROOF. See [**Spi65**, Theorem 2-5]. $\square$

DEFINITION 2.3.6. Let $\mathrm{A} \subset \mathbb{R}^m$. A function $f : \mathrm{A} \to \mathbb{R}$ is in $\mathcal{C}^k(\mathrm{A})$ if $\mathrm{D}_i f$ is in $\mathcal{C}^{k-1}(\mathrm{A})$ for all $i = 1, \cdots, m$. The set $\mathcal{C}^\infty(\mathrm{A})$ is then defined to be the set of functions such that $\mathrm{D}_i f$ is in $\mathcal{C}^k(\mathrm{A})$ for all k and for all $i = 1, \cdots, m$. Functions in this set are called **smooth** functions. In other words, f is smooth if all partials exists and are continuous. A function $f = (f^1, \cdots, f^n) : \mathrm{A} \to \mathbb{R}^n$ is smooth if each f^i is smooth for $i = 1, \cdots, n$.

REMARK 2.3.7 (Additional structure on $\mathcal{C}^\infty(\mathrm{A})$). There are two more important things to note about smooth functions which are not necessarily used in this class, but are fundamental. First, let $f, g \in \mathcal{C}^\infty(\mathrm{A})$. Then note that we can define

$$(f+g)(x) := f(x) + g(x) \qquad f \cdot g(x) := f(x) \cdot g(x).$$

The functions $f + g$ and $f \cdot g$ are both smooth. Hence $\mathcal{C}^\infty(\mathrm{A})$ forms the structure of a ring with the constant functions at 1 and 0 as the multiplicative and additive identity respectively. Furthermore, we have a compatible action of $\mathbb{R}$ on $\mathcal{C}^\infty(\mathrm{A})$: if $r \in \mathbb{R}$

$$(r \cdot f)(x) := rf(x)$$

is also a smooth function; whence $\mathcal{C}^\infty(\mathrm{A})$ is a $\mathbb{R}$-algebra. Informally, it is a $\mathbb{R}$-vector space where we know how to compatibly multiply vectors

Next, we know that the the operation of taking derivatives enjoys some very nice properties; for a list that you need to know see [**Spi65**, Theorems 2-2, 2-3]. They come under the monikers of "Liebniz rule, chain rules etc." It is reasonable to ask: can we interact with derivatives formally just knowing its properties; we state a result that uniquely characterizes the derivative (at least in one variable):

LEMMA 2.3.8. *Let* $\mathrm{A} \subset \mathbb{R}$. *Then the derivative*

$$f \mapsto f' : \mathcal{C}^\infty(\mathrm{A}) \to \mathcal{C}^\infty(\mathrm{A})$$

is characterized as the $\mathbb{R}$*-linear map satisfying the following properties:*

1. *transforms constants to zero* $1 \mapsto 0$*;*
2. *transforms linear maps to constants* $x \mapsto 1$*;*
3. *the Liebniz rule*

$$(fg)' = fg' + gf'.$$

This will be left an exercise to the reader.

In essence, the main functions of interest in differential topology are smooth functions $f : \mathbb{R}^n \to \mathbb{R}$. In fact, at this point, we can define a manifold as follows: first, let X, Y be subsets of $\mathbb{R}^n$. We say that a function $f : \mathrm{X} \to \mathrm{Y}$ is a **diffeomorphism** if f is a bijection such that f and f^{-1} are both smooth functions. Then a **smooth manifold of dimension** n is a subset $\mathrm{M} \subset \mathbb{R}^k$ such that any $x \in \mathrm{M}$ has a neighborhood $\mathrm{W} \subset \mathrm{M}$ such that W is diffeomorphic to W' where $\mathrm{W}' \subset \mathbb{R}^n$. This is a perfectly good definition except that it is rather unsatisfying: for example what does k have to do with n and it is not clear if it depends on the inclusion $\mathrm{M} \subset \mathbb{R}^k$. We want to get rid of these dependence and so we introduce the abstract concept of a smooth manifold.

3. Some recollections on topology

We run through the definitions from topology which are relevant for us; the notes for the other class gives a more extensive treatment.

3.1. Hausdorff, second-countable topological spaces.

DEFINITION 3.1.1. A **topology** on a set X is a collection $\mathcal{T} \subset \mathcal{P}(X)$ subject to the following properties:

(1) $\emptyset$ and X are in $\mathcal{T}$,
(2) $\mathcal{T}$ is closed under arbitrary unions,
(3) $\mathcal{T}$ is closed under finite intersections.

The pair $(X, \mathcal{T})$ is called a **topological space**. Elements in $\mathcal{P}(X)$ are called **open**. If $x \in X$, then any element $U \subset \mathcal{P}(X)$ such that $x \in U$ is called a **open neighborhood** of x. We often write X for a topological space (or, simply, a space) and suppress $\mathcal{T}$.

In this class, the language of topology is useful for interfacing with analysis and making "soft-analysis" argument as we had already seen. For example, the notion of continuity is easy to formulate: if X, Y are spaces and $f : X \to Y$ is a function, then it is **continuous** if for any open set $V \subset Y$, the inverse image $f^{-}(V) \subset X$ is an open set.

Often, one wants to specify a topology using only a partial collection of open sets. We have already seen this in real analysis; we begin by reviewing the idea of open balls in a slightly more abstract context.

EXAMPLE 3.1.2 (Metric spaces). In your analysis course, you might have learned the idea/definition of a metric space — perhaps at least on subsets of $\mathbb{R}^n$. Intuitively, a metric specifies a way to measure distances between points in a set. We recall that a **metric space** is a space X equipped with a function

$$d : X \times X \to \mathbb{R}$$

such that

(1) $d(x, y) \geqslant 0$ for any $x, y \in X$
(2) $d(x, x) = 0$
(3) $d(x, y) = d(y, x)$
(4) we have the triangle inequality
$$d(x, z) \leqslant d(x, y) + d(y, z).$$

Recall that we have the notion of **ϵ-balls** for $\epsilon > 0$, i.e.,

$$B(x_0, \epsilon) := \{y : d(x_0, y) < \epsilon\}.$$

These are the basic building blocks for arguments in real analysis and are the building blocks for opens in X. The notion of a basis makes this precise.

DEFINITION 3.1.3. Let X be a set. A collection $\mathcal{B} \subset \mathcal{P}(X)$ is a **basis** for a topology if:

(1) for each $x \in X$, there is at least one basis element that contains x;
(2) if x is in the intersection of two basis elements $B_1 \cap B_2$, then there is a basis element B_3 with $x \in B_3 \subset B_1 \cap B_2$.

The **topology generated by** $\mathcal{B}$ is the topology defined as follows: an set $U \subset X$ is open if for each $x \in U$ there exists an element $B \in \mathcal{B}$ such that $x \in B \subset U$. The following is a routine check and left to the reader.

LEMMA 3.1.4. *The topology generated by a basis $\mathcal{B}$ is indeed a topology. In fact, opens are exactly those which can be expressed as an arbitrary unions of elements in $\mathcal{B}$.*

EXAMPLE 3.1.5 (The Euclidean topology on $\mathbb{R}^n$). Continuing the example of metric spaces, we see that the collection $\{B(x_0, \epsilon)\}_{x_0 \in X}$ forms a basis for the topology on X; the second condition is most easily verified by drawing a picture. By Lemma 3.1.4 we see that a set $U \subset X$ is open if and only if for each $x \in U$ there exists a $\epsilon > 0$ such that $B(x_0, \epsilon) \subset U$. We will use the notation $(\mathbb{R}^n, \mathrm{Euc})$ to denote $\mathbb{R}^n$ with its usual metric topology; here Euc is meant to indicate the Euclidean topology which is the usual name for this topology.

In the other class we will learn many interesting, but perhaps pathological, topological spaces. For example, one can endow a finite set with a topology whose role seems to be combinatorial in nature. We want to only consider "geometric" topological spaces in this class and the axioms that let us do these is as follows.

Definition 3.1.6. A space X is said to be **Hausdorff** or **separated** if for any two distinct points x, y there exists open neighborhoods $x \in \mathrm{U}_x$, $y \in \mathrm{U}_y$ such that $\mathrm{U}_x \cap \mathrm{U}_y = \emptyset$. It is called **second countable** or **completely separable** if its topology has a countable basis.

Remark 3.1.7. Suppose that X is a Hausdorff space with the property that any $x \in \mathrm{X}$ admits an open neighborhood homeomorphic to an open subset of Euclidean space. Then it is second countable if and only if it is metrizable, that is, admits a metric space. We certainly want differentiable manifolds to have a notion of distance and hence second countability is desired.

At this point, we have completed Definition 2.0.1 and understood what it means for a space to have a smooth atlas. However, let us explain a quick warning.

Remark 3.1.8 (Invariance of domain). Technically, Definition 2.0.1 is a little abusive. We have to specify that our atlas is made up of open's of $\mathbb{R}^n$ for a specified n because there is no reason for this particular n to be well-defined. What allows us to get away with this abuse is a result called **invariance of domain**, proved by Brouwer using algebraic topology.

Theorem 3.1.9 (Brouwer). *Let $\mathrm{U} \subset \mathbb{R}^n$ be a open subset and $f : \mathrm{U} \to \mathbb{R}^n$ a continuous, injective function. Then $f(\mathrm{U}) \subset \mathbb{R}^n$ is open.*

Using Brouwer's theorem, we see that a space cannot have an atlas of different dimensions. Indeed, if we have a chart which constitutes a homeomorphism $f : \mathrm{U} \to \mathrm{V}$ where $\mathrm{U} \subset \mathbb{R}^k$ and $\mathrm{V} \subset \mathbb{R}^{k'}$ where $k > k'$, then we can look at the composite

$$\mathrm{U} \to \mathrm{V} \subset \mathbb{R}^{k'} \hookrightarrow \mathbb{R}^k.$$

This produces a map which is a continuous injection and thus, by Brouwer's theorem, an open map. But no open subset of $\mathbb{R}^k$ can be exclusively contained in $\mathbb{R}^{k'}$, unless it is empty.

Before we go on to define what a differentiable manifold, let us extract an important piece of data from a smooth atlas.

Remark 3.1.10 (Atlases and cocycles). The maps $\psi_{\alpha\beta} : \varphi_\alpha^{-1}(\mathrm{V}_\alpha \cap \mathrm{V}_\beta) \to \varphi_\beta^{-1}(\mathrm{V}_\alpha \cap \mathrm{V}_\beta)$, which are maps between open subsets of Eucildean space, are called **transition functions** and they satisfy

$$\psi_{\alpha\alpha} = \mathrm{id} \qquad \psi_{\alpha\beta} \circ \psi_{\beta\gamma} = \psi_{\alpha\gamma}.$$

These conditions make precise what we mean by "compatibility" and is often called the **cocycle condition**.

Furthermore, the above two conditions imply that $\psi_{\alpha\beta} \circ \psi_{\beta\alpha} = \mathrm{id}$. Hence $\psi_{\alpha\beta}$ is a **diffeomorphism**: a smooth bijection with a smooth inverse.

In fact, there is another way to specify a differentiable manifolds by glueing together cocycles; we fix n. Let us take as input a set I with elements denoted by $i \in \mathrm{I}$. For each $i \in \mathrm{I}$ we are given $\mathrm{X}_i \subset \mathbb{R}^n$ an open and for each X_i we are given an open

$$\mathrm{U}_{ij} \subset \mathrm{X}_i \qquad \forall j \in \mathrm{I}.$$

For each pair $i, j \in \mathrm{I}$ we are given diffeomorphisms

$$\psi_{ij} : \mathrm{U}_{ij} \to \mathrm{U}_{ji}$$

for which 1) $\mathrm{U}_{ii} = \mathrm{X}_i$, 2) $\psi_{ij}^{-1}(\mathrm{U}_{ji} \cap \mathrm{U}_{jk}) = \mathrm{U}_{ij} \cap \mathrm{U}_{ik}$ and the diagram

$$\begin{tikzcd}
\mathrm{U}_{ij} \cap \mathrm{U}_{ik} \arrow[rr, "\psi_{ik}"] \arrow[dr, "\psi_{ij}"] & & \mathrm{U}_{ki} \cap \mathrm{U}_{kj} \\
& \mathrm{U}_{ji} \cap \mathrm{U}_{jk} \arrow[ur, "\psi_{jk}"] &
\end{tikzcd}.$$

commutes. Then there exists topological manifold X and a smooth atlas of dimension n, $\{(\mathrm{X}_i, \mathrm{V}_i, \varphi_i)\}$ such that

(1) $\varphi_i : \mathrm{X}_i \to \mathrm{V}_i$ is a homeomorphism;
(2) $\varphi_i(\mathrm{U}_{ij}) = \mathrm{V}_i \cap \mathrm{V}_j$
(3) $\psi_{ij} = \varphi_j^{-1}|_{\mathrm{V}_i \cap \mathrm{V}_j} \circ \varphi_i|_{\mathrm{U}_{ji}}$.

This will be left as exercise in homework 2.

Now, two smooth atlases on X are said to be **compatible** if their union is an atlas and an atlas is **maximal** if it is one with the property that any atlas compatible with it must be in it.

DEFINITION 3.1.11. An n**-dimensional smooth manifold** or, simply, an n**-manifold** is a second-countable, Hausdorff space X with a maximal atlas.

In practice, one finds a smooth atlas on X and then generate a maximal one by the following lemma:

LEMMA 3.1.12. *Every smooth atlas is contained in a* unique *maximal atlas.*

PROOF. See [**Kup**, Lemma 2.2.2]. □

CHAPTER 3

Examples of smooth manifolds

We now discuss examples on smooth manifolds and how to get new smooth manifolds out of old ones. From now on, by a manifold we really mean a smooth one unless otherwise stated. We want to illustrate several methods for showing that something is a manifold. We will repeatedly use the following fact without further comment.

Lemma 0.0.1. *Let* X *be a space. If* $\mathrm{A} \subset \mathrm{X}$ *is a subset, then the subspace topology on* A *satisfies:*

(1) if X *is Hausdorff, then* A *is;*
(2) if X *is second countable, then* A *is.*

1. Spheres

Let us now give $\mathrm{S}^n \subset \mathbb{R}^{n+1}$ the structure of a smooth manifold of dimension n. Evidently, it is second countable and Hausdorff by Lemma 0.0.1 since it is a subspace of $\mathbb{R}^{n+1}$. Consider

$$\mathrm{U}_+ := \mathrm{S}^n \smallsetminus \{(1, 0, \cdots, 0)\} \qquad \mathrm{U}_- := \mathrm{S}^n \smallsetminus \{(-1, 0, \cdots, 0)\}.$$

We then define

$$\varphi_+^{-1} : \mathrm{U}_+ \to \mathbb{R}^n \qquad \tfrac{1}{1-x_0}(x_1, \cdots, x_n),$$
$$\varphi_-^{-1} : \mathrm{U}_- \to \mathbb{R}^n \qquad \tfrac{1}{x_0+1}(x_1, \cdots, x_n).$$

Indeed, φ_+^{-1} and φ_-^{-1} are diffeomorphisms which generalizes the (inverse of) the example of S^1 at the very beginning of this class.

It can be a little cumbersome to “eyeball” charts for manifolds, especially if it comes as the vanishing locus of some equation in n-space; what I mean by this is that S^n is the vanishing locus of the quadratic equation

$$q : \mathbb{R}^{n+1} \to \mathbb{R} \qquad (x_0, \cdots, x_n) \qquad x_0^2 + \cdots x_n^2 - 1,$$

and we want to say that q is nice enough such that its zero locus must have the structure of a smooth manifold. Furthermore, there is nothing special about the number 1, the equation $(x_0, \cdots, x_n) \qquad x_0^2 + \cdots x_n^2 - 57$ is also a nice space with a smooth structure and it is diffeomorphic to S^1.

However, one must be careful: not all equations will have a vanishing locus which is smooth. Clearly if we consider $x_0^2 + \cdots x_n^2$, then its vanishing locus is just the point $(0, \cdots, 0)$; it does have a smooth structure but there's something “off” because it is zero dimensional. A better class of examples is discussed in the following remark

Remark 1.0.1 (Real, punctured elliptic curves). Consider the vanishing locus of the following family of functions

$$f_b(x, y) = y^2 - x^3 - b.$$

For all $b \neq 0$, note that we have manifolds diffeomorphic to $\mathbb{R}^1$. But when $b = 0$ we get the “cusp.” This is a space, homeomorphic to $\mathbb{R}^1$ but has no smooth structure, because of the cusp at $(0, 0)$.

The following question is what we would like to answer next:

QUESTION 1.0.2. Suppose that $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is a function, when is $f^{-1}(0) \subset \mathbb{R}^{n+1}$ is an n-dimensional manifold.

Here's our strategy: as $\mathrm{Z} := f^{-1}(0)$ is already second countable and Hausdorff, we just need to produce an atlas and then take something maximal containing it. In particular, the following is enough: for each $x \in \mathrm{Z}$, find an open subset $\mathrm{U} \subset \mathrm{Z}$ such that U is diffeomorphic to a subset of $\mathbb{R}^n$. In particular, this is a local check: just choose some U first and then find a $\mathrm{U}_0 \subset \mathrm{U}$ (so, possibly after shrinking U) which is indeed diffeomorphic to a subset of $\mathbb{R}^n$.

In multivariable calculus, we have already learned a method of showing that some function is diffeomorphic, at least for a map $f : \mathbb{R}^n \to \mathbb{R}^n$.

THEOREM 1.0.3 (Inverse function theorem). *Let* $\mathrm{U} \subset \mathbb{R}^n$ *be an open subset and* $x \in \mathrm{U}$. *Assume that* $f : \mathrm{U} \to \mathbb{R}^n$ *is a smooth map whose derivative* $\mathrm{D}f_x : \mathbb{R}^n \to \mathbb{R}^n$ *is invertible. Then there is an open subset* $\mathrm{U}' \subset \mathrm{U}$ *containing* x *such that* $f(\mathrm{U}')$ *is open and* $f|_{\mathrm{U}'} : \mathrm{U} \to f(\mathrm{U})$ *is a diffeomorphism.*

Theorem 1.0.3 is plainly very useful: it converts the task of showing that f is diffeomorphism into one of linear algebra, at least if we are willing to shrink around x. The inverse function theorem can be used to study maps into $\mathbb{R}^m$ when $m \leqslant n$. To state this result, let us introduce a piece of terminology. Let $\mathrm{U} \subset \mathbb{R}^n$ be an open subset. A map $f : \mathrm{U} \to \mathbb{R}^m$ is a **submersion at** $x \in \mathrm{U}$ if the linear map $\mathrm{D}f_x : \mathbb{R}^n \to \mathbb{R}^m$ is surjective.

THEOREM 1.0.4 (Submersion theorem). *Let* $\mathrm{U}_0 \subset \mathbb{R}^n$ *be an open subset,* $x \in \mathrm{U}_0$ *and let* $\mathrm{U}_0 \xrightarrow{g} \mathbb{R}^n$ *be an open subset. Assume that* $g : \mathrm{U}_0 \to \mathbb{R}^m$ *is a smooth map and is a submersion and* $m \leqslant n$. *Then there exists an open neighborhood* $\mathrm{U} \subset \mathrm{U}_0$ *containing* x, *an open* $\mathrm{V} \subset \mathbb{R}^m$ *containing* $g(x)$ *and diffeomorphisms*

$$\psi : \mathbb{R}^n \to \mathrm{U} \qquad \varphi : \mathbb{R}^m \to \mathrm{V}$$

such that

(1) $\psi(0) = x$,
(2) $\varphi(0) = g(x)$ *and a commutative diagram*

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "\psi"] \arrow[d, "\pi"] & \mathrm{U} \arrow[d, "g"] \\ \mathbb{R}^m \arrow[r, "\varphi"] & \mathrm{V}. \end{tikzcd}$$

Here, π *is the projection map* $(x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_m)$.

Here is how we can use Theorem 1.0.8. Let $y \in g^{-1}(g(x))$; we want to find a diffeomorphism between an open subset $\mathrm{U}_1 \subset g^{-1}(g(x))$ (in the subspace topology) and an open subset of $\mathbb{R}^{n-m}$; this will then assemble (as y varies) into an atlas for $g^{-1}(g(x))$. The commutativity of the diagram in part (2) of Theorem 1.0.8 then says that we have a diffeomorphism

$$g^{-1}(g(x)) \cap g^{-1}(\mathrm{V}) \cong g^{-1}(g(x)) \cap \mathrm{U} \cong \pi^{-1}(0) = \{0\} \times \mathbb{R}^{n-m},$$

which is the desired diffeomorphism. This provides an easy way to prove that spheres are smooth.

EXAMPLE 1.0.5. We have the function $g : \mathbb{R}^{n+1} \to \mathbb{R}$ given by $(x_0, \cdots, x_n) \mapsto x_0^2 + \cdots + x_n^2$. Its derivative is given by the matrix

$$\begin{pmatrix} 2x_0 & 2x_1 & \cdots & 2x_n \end{pmatrix}.$$

This matrix has full rank if and only if $(x_0, \cdots, x_n) \neq 0$. Therefore Theorem 1.0.8 states that $g^{-1}(c)$ is a smooth manifold if and only if $c \neq 0$.

2. Projective spaces

S^n is just one of other possible ways to generalize S^1. An important class of spaces that are ubiquitous throughout mathematics are the *projective spaces.* We first introduce **real projective spaces**. The real projective space $\mathbb{RP}^n$ is the space of "lines in $\mathbb{R}^{n+1}$ through the origin." To make this precise, we consider the following equivalence relation on $\mathbb{R}^{n+1} \setminus \{0\}$:

$$(x_0, \cdots, x_n) \sim (rx_0, \cdots, rx_n) \qquad r \in \mathbb{R} \setminus \{0\}.$$

The quotient is denoted by

$$\mathbb{RP}^n := \mathbb{R}^{n+1} \setminus 0/\sim .$$

Points in $\mathbb{RP}^n$ (in other words, equivalence classes) are denoted by $[x_0 : x_1 : \cdots : x_n]$. We have the quotient map $q : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ and the inverse image over each point is a copy of $\mathbb{R} \setminus \{0\}$. It is given the quotient topology: a subset $U \subset \mathbb{RP}^n$ is open if and only if $q^{-1}(U)$ is open. One can check easily that it is second countable and Hausdorff.

The interesting part is to write down an smooth atlas for $\mathbb{RP}^n$. Consider the open subsets

$$U_i = \{[x_0 : x_1 : \cdots : x_n] : x_i \neq 0\};$$

then the maps $\varphi_i : U_i \to \mathbb{R}^n$ defined by

$$\varphi_i([x_0 : x_1 : \cdots : x_n]) = (x_1/x_i, \cdots, \widehat{x_i/x_i}, \cdots, x_n/x_i)$$

defines the (inverse) of an atlas. We also have an analog for complex space which is discussed in the homework.

3. Tori

We now discuss tori. To begin with, we note:

Lemma 3.0.1 (Products of manifolds). *Let M and N be manifolds of dimension m and n respectively. Then $M \times N$ admits the structure of an $m + n$-manifold.*

Proof. After noting that second countability and Hausdorfness are stable under products (with the product topology on $M \times N$, we need only equip M and N with an atlas. But now if $\{(U_\alpha, V_\alpha, \varphi_\alpha)\}$ is an atlas for M and $\{(U'_\beta, V'_\beta, \varphi'_\beta)\}$ is an atlas for N, then we can produce an atlas for $M \times N$ given by $\{U_\alpha \times U'_\beta, V_\alpha \times V'_\beta, \varphi_\alpha \times \varphi'_\beta)\}$. $\square$

The n-**torus** is the n-manifold given by the n-fold product of S^1's.

$$\mathbb{T}^n = S^1 \times \cdots \times S^1$$

This is yet another way to specify a manifold — by producing them out of old ones. We want to discuss a construction of the 2-torus which is somewhat "visual" and is useful in practice. Consider the "box" $I \times I$ where $I = [0, 1]$; its boundary is defined as $\partial(I \times I) := \{(x, y) : x \in \{0, 1\} \text{ or } y \in \{0, 1\}\}$. Consider the equivalence relation given by

$$(0, y) \sim (1, y) \qquad (x, 0) \sim (x, 1)$$

and $(x, y) \sim (x', y')$ if and only if $(x, y) = (x', y')$ if the points are not on the boundary. In other words, we are trying to glue the edge of the square to its opposite edge. We can give this the quotient topology: let us recall that this means, under the map:

$$q : I \times I \to I \times I/\sim$$

a set $U \subset I \times I/\sim$ is open if and only if $q^{-1}(U)$ is open. Let us endow the quotient with a smooth structure; tentatively we call this space $\mathbb{T}^2_q$ to indicate that we don't quite know that it is diffeomorphic to $I \times I$ and we will write elements in it as $[(x, y)]$ to indicate the equivalence class of (x, y).

There are several flavors of charts:

(Interior) if $(x, y) \notin q(\partial(I \times I))$ this is the easiest chart: just take a small ball around (x, y).

(Edges) For $[(x,0)]$ such that $x \neq 0, 1$, define

$$\mathrm{B}_\epsilon((x,0))(\subset \mathbb{R}^2) \to \mathbb{T}^2_q$$

given by

$$(x',y') \mapsto \begin{cases} [(x',y'+1)] & y' < 0 \\ [(x',y')] & y' \geqslant 0. \end{cases}$$

In words, if we are in the "half-moon" where $y' < 0$, translate that point by 1. The same goes for $[(0,y)]$ where $y \neq 0, 1$.

(Corner) The most interesting part is the corner: the class $[(0,0)]$ which contains $(0,0), (1,0), (1,1), (0,1)$. The idea is to split up the ball into 4 "quarter-moons." Explcitly:

$$\mathrm{B}_\epsilon(0,0) \to \mathbb{T}^2_q$$

$$(x',y') \mapsto \begin{cases} [(x'+1,y'+1)] & y' < 0, x' < 0 \\ [(x'+1,y')] & x' < 0 \\ [(x',y'+1)] & y' < 0 \\ [(x',y')] & \text{else.} \end{cases}$$

What is left to check is that the transition maps are smooth: this is the case because they are either identity of some translation, which is clearly a smooth function.

CHAPTER 4

Morphsims of manifolds, tangent spaces and derivatives

We now begin to discuss the following problem:

QUESTION 0.0.1 (Classification problem). Given an integer $n \geqslant 0$, what are all the possible manifolds of dimension?

It is clear that the only (connected) manifold of dimension 0 is the point. In dimension 1, we have $\mathbb{R}$ and S^1, but how do we know that they are different. In this case, it is quite easy to tell them apart purely topologically since one is compact and the other is not. To proceed further, we need to say what we mean for manifolds to be "the same" — evidently we need them to be the same topologically (aka homeomorphism) and we need preservation of differentiable structures.

1. Smooth maps of manifolds

Let M and N be manifolds of dimension m and n respectively, with smooth atlases $\{(\mathrm{U}_\alpha, \mathrm{V}_\alpha, \varphi_\alpha)\}_{\alpha\in\mathrm{A}}$ and $\{(\mathrm{U}'_\beta, \mathrm{V}'_\beta, \varphi'_\beta)\}_{\beta\in\mathrm{B}}$ respectively. Suppose that $f : \mathrm{M} \to \mathrm{N}$ is a continuous map. Then we say that f is **smooth** or a **morphism of manifolds** if for all $\alpha \in \mathrm{A}$ and $\beta \in \mathrm{B}$ the map

$$\varphi_\alpha : \varphi_\alpha^{-1}(\mathrm{V}_\alpha \cap f^{-1}(\mathrm{V}'_\beta)) \xrightarrow{\varphi_\alpha} \mathrm{V}_\alpha \cap f^{-1}(\mathrm{V}'_\beta) \xrightarrow{f} \mathrm{V}'_\beta \xrightarrow{\varphi_\beta^{-1}} \varphi_\beta^{-1}(\mathrm{V}'_\beta)$$

is a smooth map between open subsets of Euclidean spaces. A smooth map $f : \mathrm{M} \to \mathrm{N}$ is said to be a **diffeomorphism** if it is a continuous bijection with a smooth inverse. In this class, we want to say that two manifolds are the same whenever there is a diffeomorphism between them. In the homeworks, you will be asked to show that S^1 is diffeomorphic to $\mathbb{RP}^1$ and S^2 is diffeomorphic to $\mathbb{CP}^1$.

In the next example, we exhibit ways in which we can prove that something is a smooth map.

EXAMPLE 1.0.1 (The diagonal). A smooth map that is always defined for any manifold is the diagonal: as a continuous map of topological spaces we set

$$\Delta : \mathrm{M} \to \mathrm{M} \times \mathrm{M} \qquad x \mapsto (x, x).$$

To "officially" prove that Δ is a smooth map, one needs to pick an atlas on both domain and target. Of course the atlas on the domain does induce an atlas on the target: if $\{(\mathrm{U}_\alpha, \mathrm{V}_\alpha, \varphi_\alpha)\}_{\alpha\in\mathrm{A}}$ is an atlas then $\{(\mathrm{U}_\alpha \times \mathrm{U}_\beta, \mathrm{V}_\alpha \times \mathrm{V}_\beta, \varphi_\alpha \times \varphi_\beta\}_{\alpha,\beta\in\mathrm{A}}$ gives an atlas. However, it turns out that we do not need to pick an atlas on the target by the next lemma which asserts that smoothness is a local property.

To use the next lemma, we take a chart $(\mathrm{U}_\alpha, \mathrm{V}_\alpha, \varphi_\alpha)$ containing x and take the chart $(\mathrm{U}_\alpha \times \mathrm{U}_\alpha, \mathrm{V}_\alpha \times \mathrm{V}_\alpha, \varphi_\alpha)$. We are then left to prove that the diagonal map between subsets of Euclidean spaces:

$$\mathrm{U}_\alpha \to \mathrm{U}_\alpha \times \mathrm{U}_\alpha$$

is smooth.

The proof of the next lemma is completely analogous to the proof of the lemma in topology that a function is continuous if and only if it is continuous at each point.

LEMMA 1.0.2. *Let $f : \mathrm{M} \to \mathrm{N}$ be a continuous map of topological spaces. Then f is smooth if and only if for all $x \in \mathrm{M}$ there is a chart $(\mathrm{U}_\alpha, \mathrm{V}_\alpha, \varphi_\alpha)$ around x and a chart $(\mathrm{U}'_\beta, \mathrm{V}'_\beta, \varphi_\beta)$ around $f(x)$ such that the composite*

$$\varphi_\alpha : \varphi^{-1}(\mathrm{V}_\alpha \cap f^{-1}(\mathrm{V}'_\beta)) \xrightarrow{\varphi_\alpha} \mathrm{V}_\alpha \cap f^{-1}(\mathrm{V}'_\beta) \xrightarrow{f} \mathrm{V}'_\beta \xrightarrow{\varphi_\beta^{-1}} \varphi_\beta^{-1}(\mathrm{V}'_\beta)$$

is smooth.

Another important, albeit somewhat tautological, class of smooth maps are given by open inclusions.

LEMMA 1.0.3. *Let M be an n-manfiold, then any open subset $\mathrm{U} \subset \mathrm{M}$ is also an n-manifold. The inclusion map $i : \mathrm{U} \to \mathrm{M}$ is smooth.*

PROOF. Note that if $\{(\mathrm{U}_\alpha, \mathrm{V}_\alpha, \varphi_\alpha)\}$ is an atlas for M, then $\{(\mathrm{U}_\alpha \cap \mathrm{U}, \mathrm{V}_\alpha \cap \mathrm{U}, \varphi_\alpha)\}$ is an atlas for U. The second assertion follows from the fact that if $\mathrm{U}, \mathrm{V} \subset \mathbb{R}^n$ are opens such that $\mathrm{U} \subset \mathrm{V}$, then the inclusion map $i : \mathrm{U} \to \mathrm{V}$ is smooth. □

The next lemma asserts the standard properties of smooth functions:

LEMMA 1.0.4. *The following holds:*

(1) the identity map is smooth;
(2) smooth maps are stable under composition: if $f : \mathrm{M} \to \mathrm{N}$ and $g : \mathrm{N} \to \mathrm{P}$ is smooth, then $g \circ f : \mathrm{M} \to \mathrm{P}$ is as well;
(3) Local nature: if $\{\mathrm{U}_i\}$ is an open cover of M (a collection of open subsets U of M) such that $\cup \mathrm{U}_i = \mathrm{M}$. Then $f : \mathrm{M} \to \mathrm{N}$ is smooth if and only if $f|_{\mathrm{U}_i} : \mathrm{U}_i \to \mathrm{N}$ is a smooth map (where we have endowed U_i with the smooth structure as in Lemma 1.0.3.

PROOF. Prove it yourself or [**Kup**, Lemma 4.1.12]. □

1.1. Digression: Lie groups. Now that we know how to define smooth maps of manifolds, we can study a large and very interesting class of examples. We note that S^1 is actually a group: it is most easily described using complex coordinates. Note that $\mathrm{S}^1 = \{z : |z| = 1\} \subset \mathbb{C}$ and so any element in S^1 can be denoted by $e^{i\theta}$. The addition law is then described by addition of angles $e^{i\theta} \cdot e^{i\varphi} = e^{i(\theta+\varphi)}$. This leads to the definition of a "group in manifolds" also called a Lie group:

DEFINITION 1.1.1. A **Lie group** is a group G such that the maps

$$\mathrm{G} \times \mathrm{G} \to \mathrm{G} \qquad (x, y) \mapsto xy$$

and

$$\iota : \mathrm{G} \to \mathrm{G} \qquad x \mapsto x^{-1}$$

are smooth.

It is easy to see that S^1 is a Lie group; in fact it is an abelian Lie group: this just means that the multiplication is commutative. Furthermore, the n-torus $\mathbb{T}^n$ is also one since we know how to take products of groups and manifolds. One of the most important examples of a Lie group is

EXAMPLE 1.1.2 (General linear groups). Let $\mathrm{GL}_n(\mathbb{R})$ be the set of $n \times n$-invertible matrices. It is a manifold of dimension n^2 once we know that opens of a smooth manifold of dimension n is also one; Lemma 1.0.3. Now, we have the determinant map $\det : \mathrm{M}_n(\mathbb{R}) \to \mathbb{R}$ which is a continuous function and $\mathrm{GL}_n(\mathrm{R})$, as a set, is given by $\mathrm{M}_n(\mathbb{R}) \smallsetminus \det^{-1}(0)$ and hence is an open subset of $\mathrm{M}_n(\mathbb{R})$. The latter is isomorphic to $\mathbb{R}^{n^2}$ which does have the structure of a smooth n^2-manifold. Therefore $\mathrm{GL}_n(\mathbb{R})$ is an n^2-manifold. To check that the multiplication and inverses are smooth, we note that they can be described as polynomials in the entries which are smooth functions.

EXAMPLE 1.1.3 (Special linear groups). $\mathrm{SL}_n(\mathbb{R}) \subset \mathrm{GL}_n(\mathbb{R})$ is the subset of invertible matrices with determinant 1. In other words, it is $\det^{-1}(1)$ where $\det : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}$ is the determinant map. Using the submersion theorem, one can show that $\mathrm{SL}_n(\mathbb{R})$ is smooth of dimension $n^2 - 1$.

The next example is very important and illustrates the kind of phenomena, in simplified form, that we want to capture via the general notion of a derivative.

EXAMPLE 1.1.4. [Orthogonal groups] Let $(-,-) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the standard inner product on n-space. Explicitly this is given by

$$(x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n)) = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

In matrix multiplication, we can write this as

$$^t x \cdot y = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ \cdots \\ y_n \end{pmatrix} = (x, y).$$

The **orthogonal group** is the subset $\mathrm{O}(n) \subset \mathrm{M}_n(\mathbb{R})$ given by those matrices A such that

$$(x, y) = (\mathrm{A}x, \mathrm{A}y)$$

for all $x, y \in \mathbb{R}^n$. In other words, $\mathrm{O}(n)$ is the set of invertible matrices that preserve the inner product. One should interpret this as the set of matrices that preserve angles and lengths. Equivalently, this is the set of all matrices whose inverse is the transpose. The set $\mathrm{O}(n)$ itself admits the structure of a manifold of dimension $n(n-1)/2$.

Let's prove this because it illustrates the submersion theorem quite nicely. Let $\mathrm{S}_n(\mathbb{R}) \subset \mathrm{M}_n(\mathbb{R})$ be the subset of **symmetric matrices**: those such that $^t\mathrm{A} = \mathrm{A}$. Hopefully it is clear why we call these symmetric matrices. The first claim is that $\mathrm{S}_n(\mathbb{R})$ is a vector space of dimension $(n^2 - n)/2 + n = n(n+1)/2$ (hint: the first expression gives a hint for a proof). Indeed: this follows from the standard properties of the transpose $^{\mathrm{T}}(\mathrm{A} + \mathrm{B}) =^{\mathrm{T}} \mathrm{A} +^{\mathrm{T}} \mathrm{B}$ and $^{\mathrm{T}}(r\mathrm{A}) = r(^{\mathrm{T}}\mathrm{A})$. Therefore, it is isomorphic to $\mathbb{R}^{\frac{n(n+1)}{2}}$ as a topological space and hence a smooth manifold. We have the map

$$t : \mathrm{GL}_n(\mathbb{R}) \to \mathrm{S}_n(\mathbb{R}) \qquad \mathrm{A} \mapsto^{\mathrm{T}} \mathrm{AA}$$

and $t^{-1}(\mathrm{I}_n) = \mathrm{O}(n)$. We note that t is actually a map between Euclidean spaces!

$$t : \mathbb{R}^{n^2} \to \mathbb{R}^{\frac{n(n+1)}{2}}.$$

We want to prove that the set t is a submersion at all inverse elements in $t^{-1}(\mathrm{I}_n)$ in order to apply the submersion theorem. Hence fix $\mathrm{A} \in t^{-1}(\mathrm{I}_n)$ and we want to say that

$$\mathrm{D}t_{\mathrm{A}} : \mathbb{R}^{n^2} \to \mathbb{R}^{\frac{n(n+1)}{2}}$$

has full rank. The best way to compute that the Jacobian matrix has full rank is to straight up compute it from first principles. We claim that

$$\mathrm{D}t_{\mathrm{A}}(\mathrm{B}) = \mathrm{B}^{\mathrm{T}}\mathrm{A} +^{\mathrm{T}} \mathrm{AB}.$$

Indeed, we note that the following expression does make sense

$$\lim_{h \to 0}\left(\frac{t(\mathrm{A}+h\mathrm{B}) = t(\mathrm{A})}{h} = \frac{^{\mathrm{T}}(\mathrm{A}+h\mathrm{B})(\mathrm{A}+h\mathrm{B}) - ^{\mathrm{T}}\mathrm{AA}}{h} = \frac{^{\mathrm{T}}\mathrm{AA} + h^{\mathrm{T}}\mathrm{AB} + h\mathrm{A}^{\mathrm{T}}\mathrm{B} + (h^2)^{\mathrm{T}}\mathrm{BB} - ^{\mathrm{T}}\mathrm{AA}}{h}\right)$$

which is equal to $\mathrm{B}^{\mathrm{T}}\mathrm{A} +^{\mathrm{T}} \mathrm{AB}$ and is the desired linear transformation solving the requirement for the derivative as discussed in chapter 1. Now, to apply the submersion theorem, we need to prove that for each $\mathrm{C} \in \mathrm{S}_n(\mathbb{R})$ there indeed exists a B such that

$$\mathrm{D}t_{\mathrm{A}}(\mathrm{B}) = \mathrm{C}.$$

Using that $^{\mathrm{T}}\mathrm{AA} = \mathrm{I}_n$ by assumption, we see that $\frac{1}{2}\mathrm{CA}$ works!

2. The tangent space and the derivative

We now come to a rather tricky part of the course: defining the tangent space at a point of a manifold and the notion of the derivative of a smooth map between manifolds. The idea is actually relatively simple: suppose that that our manifolds are embedded in Euclidean space, or even (secretly) Euclidean space itself (see the example in Example 1.1.4), $f : \mathrm{M} \to \mathrm{N}$. Then for each point $x \in \mathrm{M}$ we have seen that the derivative is a certain linear map

$$\mathrm{D}f_x : \mathbb{R}^m \to \mathbb{R}^n$$

which can concretely be thought of as a Jacobian matrix. Note that the domain and the range are all of linear space even though the derivative is "local in nature" in the sense that it is defined with only reference to open neighborhoods of x. But this makes sense for manifolds because around x it is supposed to "look like" Euclidean space. Now the vector spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ should not be thought of as an "abstract" Euclidean space: they should have something to do with x and $f(x)$ (even though they will be abstractly isomorphic to $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. So we should write something like

$$\mathrm{D}f_x : \mathrm{T}_x\mathrm{M} \to \mathrm{T}_{f(x)}\mathrm{N}.$$

Furthermore, $\mathrm{D}f_x$ should look and feel like the derivative.

It is easy to be lost in the formalism of tangent spaces, but the desiderata is very clear and so we will state them here:

(1) to each $x \in \mathrm{M}$ we attach a $\mathbb{R}$-vector space $\mathrm{T}_x\mathrm{M}$;
(2) a chart around X should make an identification between $\mathrm{T}_x\mathrm{M}$ and $\mathbb{R}^n$ where n is the dimension of M — in particular this identification should rely in a meaningful way to the smooth structure of M around x;
(3) as x varies across M, the "assignment" $x \mapsto \mathrm{T}_x\mathrm{M}$ should be continuous in some precise way;
(4) $\mathrm{D}f_x$ is a $\mathbb{R}$-linear map;
(5) if f is linear (around x) then $\mathrm{D}f_x$ is just f again;
(6) if f is the identity then $\mathrm{D}f_x$ is the identity linear transformation;
(7) then $\mathrm{D}f_x$ is the identity and we should have a chain rule $\mathrm{D}(g \circ f)_x = \mathrm{D}g_{f(x)} \circ \mathrm{D}f_x$.

For practical purposes I have just outlined what the derivative should be and that is most of how you will interact with the derivative. But, of course, we should go ahead and define it properly. We will take the "algebraic approach."

2.1. Germs of functions. Recall that, in the definition of a smooth manifold, we are equipping a topological manifold with a "smooth structure." This is made precise by saying that the transition functions between charts are smooth — hence $\mathbb{R}$-valued smooth functions from one chart are all preserved. This is the starting point of one approach to the tangent space: whatever the tangent space is, maps between manifolds must induce a linear function, the derivative, between them. So the tangent space is the "domain" of derivatives and so must be some kind of " space of functions" since derivatives take in functions. To make sense of this we define germs.

Definition 2.1.1. Let M, N be a manifolds and $x \in \mathrm{M}$ be fixed. We define an equivalence relation on the set

$$\{(f, \mathrm{U}) : \mathrm{U} \text{ is an open neighborhood of } x,\ f : \mathrm{U} \to \mathrm{N} \text{ is a smooth function}\}.$$

where $f \sim g$ if and only if there exist a V, an open neighborhood of x such that $f|_{\mathrm{V}} = g|_{\mathrm{V}}$. The equivalence class of f will be denoted by $\overline{f} : (\mathrm{M}, x) \to \mathrm{N}$ and is called a **germ** or, more precisely, a **germ at** x. We also sometimes want to emphasize that $f(x) = y$ and write $\overline{f} : (\mathrm{M}, x) \to (\mathrm{N}, f(x)) = y$ for a function germ. A **function germ (at** x**)** is a germ $f : (\mathrm{M}, x) \to \mathbb{R}$ and is denoted by

$$\mathcal{O}_{\mathrm{M},x} := \{(f, \mathrm{U}) : \mathrm{U} \text{ is an open neighborhood of } x,\ f : \mathrm{U} \to \mathbb{R} \text{ is a smooth function}\}/\sim.$$

REMARK 2.1.2 (Germs and sheaves). There is a more "global" notion of germs: to each chart $(\mathrm{U}, \mathrm{V}, \varphi)$ we can associate the $\mathbb{R}$-algebra

$$\mathcal{C}^\infty(\mathrm{U}) =: \mathcal{C}^\infty(\mathrm{V}),$$

and if $(\mathrm{U}', \mathrm{V}', \varphi')$ is another chart such that $\mathrm{V} \subset \mathrm{V}'$ then we have a map of $\mathbb{R}$-algebras:

$$\mathcal{C}^\infty(\mathrm{V}) \to \mathcal{C}^\infty(\mathrm{V}')$$

given by

$$f : \mathrm{U} \to \mathbb{R} \mapsto \mathrm{U}' \xrightarrow{\varphi'} \mathrm{V}' \subset \mathrm{V} \xrightarrow{\varphi^{-1}} \mathrm{U} \xrightarrow{f} \mathbb{R};$$

we write the latter function as $f|_{\mathrm{V}'}$. In these terms, a function germ around a point $x \in \mathrm{M}$ is, informally, an equivalence class of a function $f \in \mathcal{C}^\infty(\mathrm{V})$ for V containing x and we identify f with $f|_{\mathrm{U}'}$ whenever $\mathrm{V}' \subset \mathrm{V}$ as long as $x \in \mathrm{V}'$. The language of sheaves and stalks, which we will not cover in this class, makes all of this precise.

The following is easy to verify:

LEMMA 2.1.3. *Let* $\mathrm{M}, \mathrm{N}, \mathrm{P}$ *be manifolds.*

1. *Composition of germs: L let* $f : (\mathrm{M}, x) \to (\mathrm{N}, f(x))$ *and* $g : (\mathrm{N}, g(x)) \to (\mathrm{P}, (g \circ f)(x))$ *be function germs. Then the composite, defined by*

$$\overline{g} \circ \overline{f} := \overline{g \circ f}$$

is well-defined.

2. $\mathbb{R}$*-vector space structure: for* $f, g \in \mathcal{O}_{\mathrm{M},x}$*, the operations*

$$\overline{f} + \overline{g} := \overline{f + g} \qquad r\overline{f} := \overline{\lambda f}$$

defines a $\mathbb{R}$*-vector space structure on* $\mathcal{O}_{\mathrm{M},x}$*.*

3. $\mathbb{R}$*-algebra structure: with the notation as above, multiplication*

$$\overline{f} \cdot \overline{g} := \overline{fg}$$

defines a $\mathbb{R}$*-algebra structure on* $\mathcal{O}_{\mathrm{M},x}$*.*

CONSTRUCTION 2.1.4 (Induced map on germs). Let $f : \mathrm{M} \to \mathrm{N}$ be a smooth map of manifolds and $x \in \mathrm{M}$, the **induced map on germs** is the map

$$f^* : \mathcal{O}_{\mathrm{N},f(x)} \to \mathcal{O}_{\mathrm{M},x} \qquad \overline{g} \mapsto \overline{g \circ f}.$$

Concretely, f^* takes the equivalence class of (U, g) where $g : \mathrm{U} \to \mathbb{R}$ and $f(x) \in \mathrm{U}$ to the equivalence class of $g \circ f : \mathrm{U}' \xrightarrow{f|_{\mathrm{U}'}} \mathrm{U} \to \mathbb{R}$ where U' is a neighborhood of x. It is easy to check that this is, in fact, a $\mathbb{R}$-algebra map.

Notice that the above elaboration on f^* only depends on open neighborhoods around x and $f(x)$. Hence, more generally, if we have a germ $f : (\mathrm{M}, x) \to (\mathrm{N}, f(x))$ then we have the induced map on function germs

$$f^* : \mathcal{O}_{\mathrm{N},f(x)} \to \mathcal{O}_{\mathrm{M},x} \qquad \overline{g} \mapsto \overline{g \circ f},$$

given by the same formula.

We will not really use the $\mathbb{R}$-algebra structure seriously, though we will use that this is a map of $\mathbb{R}$-vector spaces. The next proposition, albeit tautological, is key to the theory.

LEMMA 2.1.5 (Local nature of germs). *Let* M *be a manifold of dimension* n *and* $(\mathrm{U}_\alpha, \mathrm{V}_\alpha, \varphi_\alpha)$ *be a chart around* x *such that* $\varphi_\alpha(0) = x$*. Then there is an induced isomorphism of* $\mathbb{R}$*-vector spaces*

$$\mathcal{O}_{\mathrm{M},x} \xrightarrow{\varphi_\alpha^*} \mathcal{O}_{\mathbb{R}^n,0}.$$

Proof. In fact, Construction 2.1.4 satisfies the following: suppose that we have germs $f : (\mathrm{M}, x) \to (\mathrm{N}, f(x))$ and $(\mathrm{N}, f(x)) \to (\mathrm{P}, g(f(x)))$ we have precompositions

$$\mathcal{O}_{\mathrm{N},f(x)} \xrightarrow{f^*} \mathcal{O}_{\mathrm{M},x}$$

such that: 1) f^* is an $\mathbb{R}$-linear map, 2) $\mathrm{id}^* = \mathrm{id}$ and 3) $(g \circ f)^* = f^* \circ g^*$. Hence, we deduce that $(\varphi_\alpha^{-1})^*$ furnishes an inverse to φ_α^*. □

We will use Lemma 2.1.5 repeatedly to reduce a lot of results about tangent spaces to results about $\mathbb{R}^n$. We now define the tangent space; we explain why this is a reasonable definition later. Now, recall that we want a map $f : \mathrm{M} \to \mathrm{N}$ to induce a map $\mathrm{T}_x\mathrm{M} \to \mathrm{T}_{f(x)}\mathrm{N}$ of tangent spaces. As explained in Construction 2.1.4, function germs compose in the other direction:

$$\mathcal{O}_{\mathrm{N},f(x)} \to \mathcal{O}_{\mathrm{M},x} \qquad g : (\mathrm{N}, f(x)) \to \mathbb{R} \qquad g \circ f : (\mathrm{M}, x) \to \mathbb{R}.$$

So we might guess that $\mathrm{T}_x\mathrm{M}$ is the dual space of $\mathbb{R}$-linear functionals $\mathcal{O}_{\mathrm{M},x}^\vee := \mathrm{Hom}_{\mathbb{R}}(\mathcal{O}_{\mathrm{M},x}, \mathbb{R})$. This fixes the problem as we will now have a map

$$\mathcal{O}_{\mathrm{M},x}^\vee \to \mathcal{O}_{\mathrm{N},f(x)}^\vee.$$

However, this is not quite what we want: the space $\mathcal{O}_{\mathrm{M},x}^\vee$ is way too big and is not even finite dimensional. Remember that we want this to just be a $\mathbb{R}^n$ where $n = \dim(\mathrm{M})$. The right thing to do is to only consider "germs up to the first order." This is captured by the notion of a derivation.

Definition 2.1.6. A **derivation** (at $x \in \mathrm{M}$) is a $\mathbb{R}$-linear map

$$\mathrm{X} : \mathcal{O}_{\mathrm{M},x} \to \mathbb{R}$$

satisfying the "Liebniz rule"

$$\mathrm{X}(\overline{f}\overline{g}) = \mathrm{X}(\overline{f})\overline{g}(x) + \overline{f}(x)\mathrm{X}(\overline{g}(x)).$$

The tangent space $\mathrm{T}_x\mathrm{M}$ is the set $\{\mathrm{X} : \mathcal{O}_{\mathrm{M},x} \to \mathbb{R} : \mathrm{X} \text{ is a derivation}\}$.

A derivation takes in a germ and spits out a number; hence we should think of as a certain operation on functions which only depends on its germ. Furthermore, these operations only really depend on "first order information" of f. This is what the Liebniz rule says but the best way to see this is via the proof of Lemma 2.1.10.

Example 2.1.7. Let c be the germ of a constant funtion, then we claim that $\mathrm{X}(c) = 0$. By linearity, we may assume that $c = \underline{1}$, the constant function with value 1. Then:

$$\mathrm{X}(1) = \mathrm{X}(1 \cdot 1) = 2\mathrm{X}(1)$$

which means that $\mathrm{X}(1) = 0$.

Example 2.1.8. Consider $\mathrm{T}_0\mathbb{R}^n$. Then, for each $i = 1, \cdots, n$ we have the derivation

$$\tfrac{\partial}{\partial x_i} : \mathcal{O}_{\mathbb{R}^n,0} \to \mathbb{R} \qquad \overline{f} \mapsto \mathrm{D}_i f(0).$$

These are the most important kinds of derivation and they are linearly independent by evaluating them on the germs of the projections: $\pi_i : \mathbb{R}^n \to \mathbb{R}, (x_1, \cdots, x_n) \mapsto x_i$.

Example 2.1.9. Let $\overline{f}, \overline{g} : (\mathrm{M}, x) \to \mathbb{R}$ be function germs such that $\overline{f}(x) = \overline{g}(x) = 0$. Then $\mathrm{X}(\overline{f}\overline{g}) = \overline{f}(x)\mathrm{X}(\overline{g}) + \overline{g}(x)\mathrm{X}(\overline{f}) = 0 + 0 = 0$. Hence X annihilates the product of functions that vanish around x. We will use this result later.

The next lemma is a key result which verifies that $\mathrm{T}_x\mathrm{M}$ is indeed what we want.

Lemma 2.1.10. *The tangent space* $\mathrm{T}_x\mathrm{M}$ *is a* $\mathbb{R}$*-linear vector space of dimension* $\dim(\mathrm{M})$*. More precisely: let* $(\mathrm{U}, \mathrm{V}, \varphi)$ *be a chart around* x *such that* $\varphi(0) = x$ *and thus induces an isomorphism* $\mathcal{O}_{\mathrm{M},x} \xrightarrow{\varphi^*} \mathcal{O}_{\mathbb{R}^n,0}$ *as in Lemma 2.1.5. Then taking* $\mathbb{R}$*-linear dual induces an isomorphism*

$$\mathrm{T}_0\mathbb{R}^n \xrightarrow{\cong} \mathrm{T}_x\mathrm{M}$$

and the map

$$\mathbb{R}^n \to \mathrm{T}_0\mathbb{R}^n \qquad e_i \mapsto \tfrac{\partial}{\partial x_i}$$

is an isomorphism (here $\{e_i\}$ is the standard basis).

PROOF. We prove this in a few steps; for this proof we suppress the bar's above germs and declare that all functions should be thought of as their germs at x.

(1) The dual space is a vector space is a vector space. We claim that $\mathrm{T}_x\mathrm{M}$ is a sub-vector space of $\mathcal{O}_{\mathrm{M},x}^{\vee}$. To do so we just need to check the "Liebniz rule property" is closed under addition and scalar multiplication. For addition: on the one hand we have

$$(\mathrm{X}+\mathrm{Y})(fg) = \mathrm{X}(fg) + \mathrm{Y}(fg) = \mathrm{X}(f)g(x) + \mathrm{X}(g)f(x) + \mathrm{Y}(f)g(x) + \mathrm{Y}(g)f(x),$$

on the other

$$((\mathrm{X}+\mathrm{Y})(f))g(x) + ((\mathrm{X}+\mathrm{Y})(g))f(x) = \mathrm{X}(f)g(x) + \mathrm{Y}(f)g(x) + \mathrm{X}(g)f(x) + \mathrm{Y}(g)f(x).$$

Scalar multiplication is similar and left to the reader.

(2) By Lemma 2.1.5 we have that $\mathcal{O}_{\mathrm{M},x} \simeq \mathcal{O}_{\mathbb{R}^n,0}$ for some chart, normalized such that $\varphi_\alpha(0) = x$. Therefore we have an isomorphism of $\mathbb{R}$-vector spaces

$$\mathrm{T}_0\mathbb{R}^n \xrightarrow{\cong} \mathrm{T}_x\mathrm{M}.$$

So we are left to prove the result for $\mathrm{T}_0\mathbb{R}^n$. Here we can try to write down a basis for this vector space. Consider the following linear functionals

$$\tfrac{\partial}{\partial x_i} : \mathcal{O}_{\mathbb{R}^n,0} \to \mathbb{R} \qquad f \mapsto \mathrm{D}_i f(0) (= \tfrac{\partial f}{\partial x_i}(0)).$$

These linear functionals are derivations by the usual Liebniz rule and are linearly independent. We prove that they span.

(3) Let us write $x_i : \mathbb{R}^n \to \mathbb{R}$ for the i-th projection map and we think about it as the germ at 0. Let X be derivation, our claim is that

$$\mathrm{X} = \sum_{i=1}^{n} \mathrm{X}(x_i)\tfrac{\partial}{\partial x_i}.$$

Let's unpack what this means: this means that for all function germ f, $\mathrm{X}(f)$ is the sum of terms

$$\mathrm{X}(x_i)\tfrac{\partial f}{\partial x_i}(0) = \mathrm{X}(x_i)\mathrm{D}_i f(0).$$

where x_i is the germ of the projection function to the i-variable and $\mathrm{D}_i f(0)$ is a number: it is the value of the derivation explained in Example 2.1.8.

(4) To do so, recall that we have Taylor's theorem with remainder: we can write any smooth function f locally around 0 (in other words, valid for $x \in \mathrm{U}_0$ a small enough neighborhood of 0) as

$$f(x = (x_1, \cdots, x_n)) = f(0) + \sum_{i=1}^{n} \tfrac{\partial f}{\partial x_i}(0)x_i + \mathrm{R}_2(x)$$

where $\mathrm{R}_2(x)$ is the remainder term:

$$\mathrm{R}_2(x) = \sum_{i,j}^{n} x_i x_j \int_0^1 (1-t)\tfrac{\partial^2 f(tx)}{\partial x_i \partial x_j} dt.$$

It almost doesn't matter what $\mathrm{R}_2(x)$ is — the only thing we ever need is to be able to know that each summand can be written as a product of two functions vanishing at 0 so as to invoke Example 2.1.9.

(5) Now, we calculate:

$$\begin{aligned} \mathrm{X}(f) &= \mathrm{X}(f(0)) + \mathrm{X}(\sum_{i=1}^{n} \tfrac{\partial f}{\partial x_i}(0)x_i) + \mathrm{X}(\mathrm{R}_2) \\ &= \mathrm{X}(\sum_{i=1}^{n} \tfrac{\partial f}{\partial x_i}(0)x_i) \\ &= \sum_{i=1}^{n} \tfrac{\partial f}{\partial x_i}(0)\mathrm{X}(x_i). \end{aligned}$$

The first equality comes from plugging in the Taylor expansion and the linearity of X, the second equality comes from Examples 2.1.7 and 2.1.9 and the last equality comes from linearity.

□

Lemma 2.1.10 tells us that, in practice, the tangent space at a point $x \in \mathrm{M}$ is an n-dimensional vector space whose bases are given by the linear functions $\frac{\partial}{\partial x_i}$. This is useful for performing explicit calculations. As promised, we define the tangent space in order to get to the derivative of a smooth function between smooth manifolds; more generally, the germ of a smooth function. The definition is relatively easy: first given a germ

$$f : (\mathrm{M}, x) \to (\mathrm{N}, f(x))$$

we get an induced map (in the opposite direction) on function germs

$$f^* : \mathcal{O}_{\mathrm{N},f(x)} \to \mathcal{O}_{\mathrm{M},x}$$

which induces a map on functionals

$$(f^*)^\vee := f_* : \mathcal{O}_{\mathrm{M},x}^\vee \to \mathcal{O}_{\mathrm{N},f(x)}$$

We note that f_* preserves derivation, i.e., if X is a derivation then $f_*\mathrm{X}$ is a derivation. Indeed: if $g, g' : (\mathrm{N}, f(x)) \to \mathbb{R}$ are function germs then

$$(f_*\mathrm{X})(gg') = \mathrm{X}(gg' \circ f) = \mathrm{X}(g \circ f \cdot g' \circ f) = g(f(x))\mathrm{X}(g' \circ f) + g'(f(x))\mathrm{X}(g \circ f).$$

We define the derivative this way.

Construction 2.1.11 (Induced maps on tangent spaces). Let $f : \mathrm{M} \to \mathrm{N}$ be a smooth map of manifolds and let $x \in \mathrm{M}$. Then the **derivative at** x is given by

$$\mathrm{D}f_x : \mathrm{T}_x\mathrm{M} \to \mathrm{T}_{f(x)}\mathrm{N}$$

given by

$$\mathrm{X} \mapsto f_*\mathrm{X} = (\overline{g} \in \mathcal{O}_{\mathrm{N},f(x)} \mapsto \mathrm{X}(\overline{g} \circ \overline{f})).$$

We verify the properties of the derivation.

Lemma 2.1.12. *Let $f : \mathrm{M} \to \mathrm{N}$ be a smooth map of manifolds and let $x \in \mathrm{M}$, then*

$$\mathrm{D}f_x : \mathrm{T}_x\mathrm{M} \to \mathrm{T}_{f(x)}\mathrm{N}$$

satisfies the following properties:

(1) the map $\mathrm{D}f_x$ is an $\mathbb{R}$-linear map;
(2) if $f = \mathrm{id}$, then $\mathrm{Did}_x = \mathrm{id}$;
(3) if $g : \mathrm{N} \to \mathrm{P}$ is another smooth map then

$$\mathrm{D}(g \circ f)_x = \mathrm{D}g_{f(x)} \circ \mathrm{D}f_x.$$

(4) Let $(\mathrm{U}, \mathrm{V}, \varphi)$ *and* $(\mathrm{U}', \mathrm{V}', \varphi')$ *be charts around* x *and* $f(x)$ *respectively such that* $\varphi(0) = x, \varphi'(0) = f(x)$*, then the composite*

$$\begin{array}{ccc} \mathrm{T}_x\mathrm{M} & \xrightarrow{\mathrm{D}f_x} & \mathrm{T}_{f(x)}\mathrm{N} \\ {\scriptstyle \mathrm{D}\varphi_0, \cong}\uparrow & & \downarrow{\scriptstyle (\mathrm{D}\varphi_0')^{-1}, \cong} \\ \mathrm{T}_0\mathbb{R}^m & & \mathrm{T}_0\mathbb{R}^n \end{array}$$

coincides with the total derivative.

Proof. Properties (1)-(3) all hold for the composition

$$f_* : \mathcal{O}_{\mathrm{M},x}^\vee \to \mathcal{O}_{\mathrm{N},f(x)}^\vee$$

by the formal properties of taking duals of linear maps. Thus, it holds for $\mathrm{D}f_x$ because f_* preserves derivatives. Now we explicitly want to compute the composition

$$(\mathrm{D}\varphi_0')^{-1} \circ \mathrm{D}f_x \circ \mathrm{D}\varphi_0 = (\varphi_0'^{-1} \circ f \circ \varphi_0)_*.$$

where the equality follows from (1)-(3). So let us rewrite

$$g := \varphi_0'^{-1} \circ f \circ \varphi_0 : (\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$$

as $g = (g^1, \cdots, g^n)$ where each $g^i : \mathbb{R}^m \to \mathbb{R}$. Since we have already shown that $\frac{\partial}{\partial x_j}$ for $j = 1, \cdots m$ forms a basis for $\mathrm{T}_0\mathbb{R}^m$ we need only calculate

$$g_*(\frac{\partial}{\partial x_j})$$

in terms of the bases $\frac{\partial}{\partial y_i}$ for $i = i, \cdots, n$. In fact, to prove that g_* is indeed the usual total derivative, we need to know the coefficient corresponding to $\frac{\partial}{\partial y_i}$ is exactly the number $\frac{\partial g^i}{\partial x_j}(0)$; in the basis $(\frac{\partial}{\partial x_j})_{j=1,\cdots,m}$ for the source and $(\frac{\partial}{\partial y_i})_{i=1,\cdots n}$ we will have written out the Jacobian of g.

So take a function germ $h : (\mathbb{R}^n, 0) \to \mathbb{R}$. Then, for any fixed $j \in \{1, \cdots m\}$ we have

$$\begin{aligned} g_*(\frac{\partial}{\partial x_j})(h) &= \frac{\partial(h \circ g)}{\partial x_j} \\ &= \sum_{i=1}^n \frac{\partial h}{\partial y_i}(0) \frac{\partial g^j}{\partial x_j}(0), \end{aligned}$$

where the first equality is by definition and the second equality comes from the multivariable chain rule. Since this is true for all function germ h, we conclude that the coefficient the term $\frac{\partial}{\partial y_i}$ is as desired. □

Remark 2.1.13 (Tangent space as curves). We remark on another, equivalent way of defining the tangent space. It is somehow more "visual" and evokes the idea that a tangent vector should be a point on the manifold and a "vector pointing parallel to it." Let $(-1, 1) := \mathrm{I}$ be the open interval and we think of it as a smooth manifold. A **smooth curve around** $x \in \mathrm{M}$ (or simply a **curve** if the context is clear) is a smooth map $\gamma : \mathrm{I} \to \mathrm{M}$ such that $\gamma(0) = x$.

A smooth curve in M can be rather wild, but we are only interested in the first order information derived from γ. To do so we associate to a curve γ a derivation given by:

$$\mathrm{X}_\gamma : \mathcal{O}_{\mathrm{M},x} \to \mathbb{R} \qquad g \mapsto (g \circ \gamma)'(0).$$

In other words, we look at $g \circ \gamma : \mathrm{I} \to \mathbb{R}$ and just take its derivative at 0. By the usual Liebniz and chain rules, this is a derivation. Set

$$\Gamma_x\mathrm{M} := \{\gamma : \gamma \text{ is smooth curve around } x\}$$

and define an equivalence relation by

$$\gamma \sim \gamma' \Leftrightarrow (f \circ \gamma)'(0) = (f \circ \gamma)'(0) \forall \overline{f} \in \mathcal{O}_{\mathrm{M},x}.$$

Then one can check that $\mathrm{V}_x\mathrm{M} := \Gamma_x\mathrm{M}/\sim$ is a $\mathbb{R}$-vector space and that the map

$$\mathrm{V}_x\mathrm{M} \to \mathrm{T}_x\mathrm{M} \qquad \overline{\gamma} \mapsto \mathrm{X}_\gamma$$

is an isomorphism of $\mathbb{R}$-vector spaces.

2.2. Vector bundles and tangent bundles. We now want to assemble the tangent spaces into a global object. The idea of the tangent space is simple once we have granted ourselves the Whitney embedding theorem or we had started the class with manifolds embedded in Euclidean space. Let M be a manifold and suppose that it comes as a subset of $\mathbb{R}^{\mathrm{N}}$: so we have $\mathrm{M} \subset \mathbb{R}^{\mathrm{N}}$. Informally speaking, $\mathrm{T}_x\mathrm{M}$ is the n-dimensional affine plane in $\mathbb{R}^{\mathrm{N}}$ through x which is the best linear approximation. So $\mathrm{T}_x\mathrm{M}$ is the translate of a n-dimensional subspace of $\mathbb{R}^{\mathrm{N}}$. With this definition, we set

$$\mathrm{TM} := \{(x, v) : v \in \mathrm{T}_x\mathrm{M}\} \subset \mathbb{R}^{\mathrm{N}},$$

and one can show that this subspace of $\mathbb{R}^{\mathrm{N}}$ is a smooth manifold.

Example 2.2.1. Let $\mathrm{X} = \mathrm{S}^n$ embedded in $\mathbb{R}^{n+1}$. Then the tangent space at $x \in \mathrm{S}^n$ is given by

$$\mathrm{T}_x\mathrm{S}^n \cong \{x + v : v \perp x\} \subset \mathbb{R}^{n+1}.$$

Since we will not really use the above "embedded" definition of the tangent space, we will not bother making the idea precise. Instead, we will make the tangent bundle precise. To do so, we will put in the context of the general theory of **vector bundles**

Definition 2.2.2. Let X be a space. Then a **real vector bundle of rank** n is a topological space $\mathcal{E}$ equipped with a surjective continuous map $p : \mathcal{E} \to \mathrm{X}$ subject to the following condition:

(1) the fibre $p^{-1}(x)$ has the structure of an n-dimensional $\mathbb{R}$-vector space;
(2) for each $x \in \mathrm{M}$ there exists a neighborhood U of x and a homeomorphism

$$\Phi : p^{-1}(\mathrm{U}) \xrightarrow{\cong} \mathrm{U} \times \mathbb{R}^n$$

such that: $p_{\mathrm{U}} \circ \Phi = p$ where $p_{\mathrm{U}} : \mathrm{U} \times \mathbb{R}^n \to \mathrm{U}$ is the projection;
(3) for each $p \in \mathrm{U}$, the restriction of Φ to

$$\Phi : p^{-1}(x) \to \{x\} \times \mathbb{R}^n$$

is a $\mathbb{R}$-linear isomorphism.

If X and $\mathcal{E}$ are smooth manifolds and p is a smooth function, then we say that q is a smooth vector bundle.

Remark 2.2.3. There is also a notion of a **complex vector bundle** where we replace $\mathbb{R}$ above with $\mathbb{C}$. The analog of smoothness in this setting is to ask that X and $\mathcal{E}$ are both complex manifolds and p is holomorphic. Sometimes vector bundles have an additional structure: for example each fibre could be equipped with a nondegenerate bilinear form.

Some terminology surrounding Definition 2.2.2 we say that $\mathcal{E}$ is the **total space** of the vector bundle and p is the **projection map**.

Example 2.2.4. Let $\mathcal{E} := \mathrm{X} \times \mathbb{R}^n$, then the projection onto X map $\mathcal{E} \to \mathrm{X}$ is a vector bundle. If X is a smooth manifold of dimension m, then $\mathcal{E}$ is a smooth manifold of dimension $n + m$. This bundle is called the **trivial bundle** or sometimes called the **product bundle**.

Example 2.2.5. Consider $\mathrm{X} = \mathbb{RP}^n$; a point $x = [x_0 : \cdots : x_n]$ defines a line, which we denote by $x \subset \mathbb{R}^{n+1}$ by considering the span of $(x_0, \cdots, x_n)$; in this way we think of X as the space of lines in $n + 1$-Euclidean space. Consider

$$\mathrm{Taut}_n := \{(x, v) : v \in x\} \subset \mathrm{X} \times \mathbb{R}^{n+1}.$$

Then $\mathrm{Taut}_n \subset \mathrm{X} \times \mathbb{R}^{n+1} \xrightarrow{p_{\mathrm{X}}} \mathrm{X}$ is a smooth vector bundle of rank 1; this is called the **tautological bundle** on real projective space.

Now, we construct the most important example of a vector bundle.

CONSTRUCTION 2.2.6 (The tangent bundle). Let M be a manifold of dimension n. Begin by considering the set

$$\mathrm{TM} := \sqcup_{x \in \mathrm{M}} \mathrm{T}_x\mathrm{M}.$$

Elements can be thought of as pairs (x, v) where $x \in \mathrm{M}$ and $v \in \mathrm{T}_x\mathrm{M}$. We have the map $p : \mathrm{TM} \to \mathrm{M}$ that sends (x, v) to x.

(1) For each $x \in \mathrm{M}$ we take a chart $(\mathrm{U}, \mathrm{V}, \varphi)$. What we then do is define

$$\widetilde{\varphi} : p^{-1}(\mathrm{V}) \to \mathbb{R}^{2n} \qquad (x, v = \sum_{i=1}^{n} v^i \tfrac{\partial}{\partial x_i}) \mapsto (\varphi^{-1}(x), v^i).$$

The image is given by $\varphi^{-1}(\mathrm{V}) \times \mathbb{R}^n = \mathrm{U} \times \mathbb{R}^n$ which is an open subset of $\mathbb{R}^{2n}$. Note that this is a bijection onto its image by using φ.

(2) Given two charts $(\mathrm{U}, \mathrm{V}, \varphi)$ and $(\mathrm{U}', \mathrm{V}', \psi')$ we have diffeomorphisms

$$\mathrm{U} \cap \mathrm{U}' \times \mathbb{R}^n = \widetilde{\varphi}(p^{-1}(\mathrm{V}) \cap p^{-1}(\mathrm{V}')) \to (\widetilde{\varphi'})(p^{-1}(\mathrm{V}') \cap p^{-1}(\mathrm{V})) = \mathrm{U}' \cap \mathrm{U} \times \mathbb{R}^n$$

where the map is given by $\widetilde{\psi} \circ \widetilde{\varphi}^{-1}$.

(3) Now, pick a countable atlas $\{(\mathrm{U}_i, \mathrm{V}_i, \varphi_i)\}$ of M and consider the data of $\mathrm{U}_i \times \mathbb{R}^n \subset \mathbb{R}^{2n}$, and for each i, $\mathrm{U}_{ij} := (\mathrm{U}_i \cap \mathrm{U}_j) \times \mathbb{R}^n$ and the maps $\mathrm{U}_{ij} \to \mathrm{U}_{ji}$ as defined in part (2). One checks that this satisfies the condition of the "cocyles" explained in homework 2. Hence TM assembles into a manifold of dimension $2n$.

(4) Locally the function p is given by the projection $\mathrm{U} \times \mathbb{R}^n \to \mathrm{U}$ and thus it is smooth.

Lastly, we note the following "functoriality" property of the tangent bundle which follows easily from the construction of the tangent bundle.

LEMMA 2.2.7. *Let $f : \mathrm{M} \to \mathrm{N}$ be a map of smooth manifolds, then there is a smooth map $\mathrm{D}f$ making the following diagram commute:*

$$\begin{tikzcd} \mathrm{TM} \arrow[r, "\mathrm{D}f"] \arrow[d] & \mathrm{TN} \arrow[d] \\ \mathrm{M} \arrow[r, "f"] & \mathrm{N} \end{tikzcd}$$

such that, for each $x \in \mathrm{X}$ the map

$$\mathrm{T}_x\mathrm{M} \to \mathrm{T}_{f(x)}\mathrm{N}$$

is the map of Lemma 2.1.12.

PROOF. Note that the commutativity of the diagram says that $\mathrm{D}f(x, v) = (f(x), \mathrm{D}f_x(v))$ so the map $\mathrm{D}f$ is constructed this way by defining it on the underlying set and checking that it glues together to a map $\mathrm{D}f : \mathrm{TM} \to \mathrm{TN}$. The only thing we really need to explain is why $\mathrm{D}f$ is smooth. The point here is that

$$\mathrm{D}f(x, v = (v_1, \cdots, v_n)) = (f(x) = (f^1(x), \cdots, f^n(x)), \tfrac{\partial f}{\partial x_i}(x)v_i).$$

which is evidently smooth since f was. □

It then follows that the following global version of Lemma 2.1.12 holds:

COROLLARY 2.2.8. *The map*

$$\mathrm{D}f : \mathrm{TM} \to \mathrm{TN}$$

satisfies the following properties:

(1) if $f = \mathrm{id}$, then $\mathrm{D}f = \mathrm{id}$;

(2) if $g : \mathrm{N} \to \mathrm{P}$ is another smooth map then

$$\mathrm{D}(g \circ f)_x = \mathrm{D}g \circ \mathrm{D}f$$

coincides with the total derivative.

3. Application: globalizing calculus

We now see some of the formalism above in action. To do so, we have the following terminology which makes things convenient o state.

Definition 3.0.1. Let $f : \mathrm{M} \to \mathrm{N}$ be a smooth map of manifolds. We say that:

(1) $x \in \mathrm{M}$ is a **regular point** of f if $\mathrm{D}f_x$ is surjective;
(2) $y \in \mathrm{N}$ is a **regular value** if any $x \in f^{-1}(y)$ is a regular point;
(3) x is a **critical point** if $\mathrm{D}f_x$ is not surjective and $y = f(x)$ is a **critical value** in this case.

For the rest of this section, when we write M we always mean a smooth m-dimensional manifold and write N for a smooth n-dimensional manifold. We will prove the following three theorems.

Theorem 3.0.2 (Global inverse function theorem). *Let $f : \mathrm{M} \to \mathrm{N}$ be a bijective smooth map such that for all $x \in \mathrm{M}$, the derivative $\mathrm{D}f_x$ is bijective. Then f is a diffeomorphism.*

Theorem 3.0.3 (Global submersion theorem). *Let $f : \mathrm{M} \to \mathrm{N}$ be a smooth map and $y \in \mathrm{N}$ a regular value, then $f^{-1}(y)$ is a smooth $m - n$-dimensional manifold.*

Theorem 3.0.4 (Stack of records theorem). *Let $f : \mathrm{M} \to \mathrm{N}$ be a smooth map and assume that* $\dim(\mathrm{M}) = \dim(\mathrm{N})$. *Further assume that M is compact. For any $y \in \mathrm{N}$ which is a regular value, then:*

(1) $f^{-1}(y)$ is the finite set, $f^{-1}(y) = \{x_1, \cdots, x_n\} \subset \mathrm{M}$.
(2) there is an open neighborhood U of y such that $f^{-1}(\mathrm{U}) = \mathrm{V}_1 \sqcup \cdots \sqcup \mathrm{V}_n$ with $x_i \in \mathrm{V}_i$ and $f|_{\mathrm{V}_i} : \mathrm{V}_i \to \mathrm{U}$ is a diffeomorphism.

The three theorems have the same theme: "define globally and prove locally." This will be evident in the proofs. We also remark that Theorem 3.0.4 introduces a new condition that one would like to impose when classifying manifolds: namely that of compactness. We will use the following standard results about compactness that will be proved in the other class. Since we have restricted ourselves to studying manifolds (as opposed to arbitrary topological spaces), the notion compactness is the familiar one from analysis.

Proposition 3.0.5. *Let X be a space:*

(1) $\mathrm{X} \subset \mathbb{R}^n$ is compact if and only if X is closed and bounded;
(2) if X is compact and $\mathrm{A} \subset \mathrm{X}$ is closed then A is compact;
(3) if X is a Hausdorff, then any compact set is closed;
(4) the image of a compact set under a continuous function is compact.

Example 3.0.6. Let X be a connected 1-manifold. We will soon see that X is diffeomorphic to either $\mathbb{R}$ or S^1. In the former, X is not compact, but in the latter X is. Hence the only compact one manifolds are disjoint union of spheres.

3.1. Proofs of theorems.

We begin with Theorem 3.0.2.

Lemma 3.1.1. *Let $f : \mathrm{M} \to \mathrm{N}$ be a smooth map and $x \in \mathrm{M}$. Assume that $\mathrm{D}f_x : \mathrm{T}_x\mathrm{M} \to \mathrm{T}_{f(x)}\mathrm{N}$ is bijective. Then there exists an open neighborhood V_0 of x such that $f|_{\mathrm{V}_0} : \mathrm{V}_0 \to f(\mathrm{V}_0)$ is a diffeomorphism.*

PROOF. Using Lemma 1.0.2, a smooth map around x can be described in terms of $(\mathrm{U}, \mathrm{V}, \varphi)$ a chart around x, $(\mathrm{U}', \mathrm{V}', \varphi')$ a chart around $f(x)$ (normalized such that $\varphi(0) = x$ and $\varphi'(0) = f(x)$) and the composite (which we rename as g)

$$g : \varphi^{-1}(\mathrm{V} \cap f^{-1}(\mathrm{V}')) \xrightarrow{\varphi} \mathrm{V} \cap f^{-1}(\mathrm{V}') \xrightarrow{f} \mathrm{V}'_\beta \xrightarrow{\varphi'^{-1}} \varphi^{-1}(\mathrm{V}').$$

Note that g is a smooth function from an open of $\mathbb{R}^m$ to an open of $\mathbb{R}^n$. Applying Lemma 2.1.12 we identify $\mathrm{D}g_0$ with the usual total derivative. The inverse function theorem then gives us an open subset $\mathrm{U}_0 \subset \varphi^{-1}(\mathrm{V} \cap f^{-1}(\mathrm{V}'))$ such that $g|_{\mathrm{U}_0} : \mathrm{U}_0 \to g(\mathrm{U}_0)$ is a diffeomorphism. But now, setting charts $(\mathrm{U}_0, \varphi(\mathrm{U}_0), \varphi)$ around x and $(g(\mathrm{U}_0) \cap \mathrm{V}', \varphi'(\mathrm{U}_0 \cap \mathrm{V}'), \varphi')$ as a chart around $f(x)$, we conclude that the map $f|_{\mathrm{V}_0} : \mathrm{V}_0 \to f(\mathrm{V}_0)$ is a diffeomorphism.

□

PROOF OF THEOREM 3.0.2. Let f^{-1} be the set-theoretic inverse of f. We want to prove that f^{-1} is smooth. Let $x \in \mathrm{M}$ and consider $f(x)$; we want to prove that f^{-1} is smooth around $f(x)$. But now, by Lemma 3.1.1 and the fact that inverses are unique, f^{-1} coincides with $(f|_{\mathrm{V}_0})^{-1}$ which is smooth because $f|_{\mathrm{V}_0}$ is a diffeomorphism. □

PROOF OF THEOREM 3.0.3. Using the submersion theorem, this is proved in a similar was as in the global inverse function theorem. We will leave it to the reader.

Instead, we make several observations about how $f^{-1}(y)$ looks like around each point; say $x \in f^{-1}(y)$. Given a chart $(\mathrm{U}, \mathrm{V}, \varphi)$ of $x \in \mathrm{M}$, then the submersion theorem gives a chart of $f^{-1}(y)$ around x $(\mathrm{U}', \mathrm{V}', \varphi')$ where $\mathrm{U} = \mathrm{U} \cap \{(0, \cdots, 0, x_{n-m+1}, \cdots, x_n)\}$. Furthermore, the tangent space of x as a point of the manifold $f^{-1}(y)$ is given by

$$\mathrm{T}_x f^{-1}(y) = \ker(\mathrm{D}f_x : \mathrm{T}_x\mathrm{M} \to \mathrm{T}_y\mathrm{N}).$$

□

We now come to the proof of the "stack of records" theorem which shows how the idea of a smooth function between smooth manifolds constrains the topology of continuous functions between them. In topology we have the following concept:

DEFINITION 3.1.2. A **covering space** is a continuous surjection $p : \mathrm{E} \to \mathrm{B}$ such that every $b \in \mathrm{B}$ is **evenly covered**: there exists an open neighborhood U of b such that $p^{-1}(\mathrm{U})$ is a disjoint union of open subsets V_α of E such that $p|_{\mathrm{V}_\alpha} : \mathrm{V}_\alpha \to \mathrm{U}$ is a homeomorphism. In this case we say that E is the **total space** and B is the **base space**.

One should visualize them as "stacks of pancakes or records." Covering spaces are ubiquitous throughout mathematics and one of the key points is that they are easy to classify, relative to B. Roughly speaking, the way that covers permute completely classify covering spaces with base B. Theorem 3.0.4 asserts that maps between manifolds of the same dimension are "almost covering spaces" — the failure only happens at points which are not regular.

The idea of the proof is simple: by the global submersion theorem the inverse image of a regular value must be a zero dimensional manifold. Now, it lives in a compact space since M is assumed to be compact and the only possibility is finitely many points. Now draw neighborhoods around each point and, because of the inverse function theorem, we are allowed to shrink all these neighborhoods simulatenously such that each "layer" maps diffeomorphically onto a common image.

PROOF. We prove this in a two steps:

(1) First we claim that $f^{-1}(y)$ is a discrete space. To do so, we claim that for each $x, x' \in f^{-1}(y)$ where $x \neq x'$ there exists an opens $\mathrm{V}, \mathrm{V}' \subset \mathrm{M}$ such that $x \in \mathrm{V}$ and $x' \in \mathrm{V}$ but $\mathrm{V} \cap \mathrm{V}' = \emptyset$. This means that x and x' are open in the subspace topology and thus are singletons are open, leading us to conclude that it must be discrete. To see this: since $\mathrm{D}f_x$ is a surjective map between vector spaces of the same dimension, it is an isomorphism. Therefore, by Lemma 3.1.1 there exists an open $\mathrm{V} \subset \mathrm{M}$ such that

$f|_{\mathrm{V}} : \mathrm{V} \to f(\mathrm{V})$ is a diffeomorphism. The same holds of x' to produce an open V'. We claim that $\mathrm{V} \cap \mathrm{V}' = \emptyset$. Otherwise, $f|_{\mathrm{V}\cap\mathrm{V}'} : \mathrm{V}\cap\mathrm{V}' \to f(\mathrm{V})\cap f(\mathrm{V}')$ is a diffeomorphism. Now, $y \in f(\mathrm{V})\cap f(\mathrm{V}')$ and $f(x) = y = f(x')$. Since f is a diffeomorphism on $\mathrm{V}\cap\mathrm{V}'$ it is, in particular, injective and thus we must have that $x = x'$ which is a contradiction.

(2) Next, we claim that $f^{-1}(y)$ is in fact finite. To do so, we note that $f^{-1}(y) \subset \mathrm{M}$ is a closed subset which is furthermore discrete. But M is compact and closed subsets of compact sets are compact and thus $f^{-1}(y)$ must be finite.

(3) Now, we need to find a small enough open around $f(y)$ such that f has the covering space property onto y. List out:

$$f^{-1}(y) = \{x_1, \cdots, x_n\}.$$

For each i there exists neighborhoods W_i such that $x_i \in \mathrm{W}_i$ and $f|_{\mathrm{W}_i} : \mathrm{W}_i \to f(\mathrm{W}_i) =: \mathrm{U}_i$ is a diffeomorphism. Set

$$\mathrm{U} := \mathrm{U}_1 \cap \cdots \cap \mathrm{U}_n \smallsetminus f(\mathrm{X} \smallsetminus (\mathrm{W}_1 \cup \cdots \mathrm{W}_n)).$$

This is the intersection of all the images of W_i with points such that f is possibly not a diffeomorphism removed.

(4) We claim that U is open. Indeed, $\mathrm{U}_1 \cap \cdots \cap \mathrm{U}_n$ is open and $\mathrm{X} \smallsetminus (\mathrm{W}_1 \cup \cdots \mathrm{W}_n)$ is closed. We will be done if $f(\mathrm{X} \smallsetminus (\mathrm{W}_1 \cup \cdots \mathrm{W}_n))$ is closed. To ensure this, note that $\mathrm{X} \smallsetminus (\mathrm{W}_1 \cup \cdots \mathrm{W}_n)$ is a closed of a compact and hence compact and any continuous function preserves compact sets. Therefore $f(\mathrm{X} \smallsetminus (\mathrm{W}_1 \cup \cdots \mathrm{W}_n))$ is compact because any compact subset of a Hausdorff space is closed.

(5) By design, $y \in \mathrm{U}$ and setting $\mathrm{V}_i := \mathrm{W}_i \cap f^{-1}(\mathrm{U})$ we get that V_i's are disjoint, contains x_i and $f|_{\mathrm{V}_i}$ is a diffeomorphism onto imts image.

□

LEMMA 3.1.3. *Let f be as in the stack of records theorem. Let* $\mathrm{RegVal(f)} \subset \mathrm{N}$ *be the subset of regular values of f. Then it is an open subset and, in particular, a smooth manifold of dimension n.*

PROOF. First, we claim that the set of regular points is an open subset of M. Indeed, by Lemma 3.1.1 as soon as $\mathrm{D}f_x$ is surjective (and hence an isomorphism by dimension reasons), f is a diffeomorphism on an open neighborhood of x. Therefore, the set of regular points, call it R, is open in M. Since M is compact, $\mathrm{M} \smallsetminus \mathrm{R}$ is compact since it is closed. Therefore $f(\mathrm{M} \smallsetminus \mathrm{R})$ is a compact subset of N, whence closed. The set RegVal(f) is precisely the complement of $f(\mathrm{M} \smallsetminus \mathrm{R})$ and hence is open.

□

PROPOSITION 3.1.4. *Let f be as in the stack of records theorem. The function*

$$\mathrm{RegVal(f)} \mapsto \mathbb{N} \qquad y \mapsto |f^{-1}(y)|$$

is locally constant: for each $y \in \mathrm{RegVal(f)}$ *there exists an open subset* U *in* RegVal(f) *such that* $|f^{-1}(y')| = |f^{-1}(y)|$ *for all* $y' \in \mathrm{U}$.

PROOF. There are two cases. First, assume that $f^{-1}(y)$ is empty; this means that $y \notin f(\mathrm{M})$. Now, M is compact and hence $f(\mathrm{M}) \subset \mathrm{N}$ is compact whence closed. Therefore $\mathrm{N} \smallsetminus f(\mathrm{M})$ is open. Since $f^{-1}(y) \in \mathrm{N} \smallsetminus f(\mathrm{M})$ we are done.

Now, assume that $f(y) \neq \emptyset$. By Lemma 3.1.3, the set of regular values is open. Hence we fix $y \in \mathrm{RegVal(f)}$. By the stack of records theorem, we can find a U open neighborhood such that $y \in \mathrm{U} \subset \mathrm{RegVal}(f)$ and $f^{-1}(\mathrm{U})$ is a disjoint union $\mathrm{V}_1 \sqcup \cdots \sqcup \mathrm{V}_n$. Since f restricted to each V_i is a diffeomorphism, for any other element $y' \in \mathrm{U}$ only one point in each V_i can map to y_i. Therefore, $|f^{-1}(y')| = |f^{-1}(y)|$ as desired.

□

3.2. Application: Milnor's proof of the fundamental theorem of algebra. We now see how the concepts that we have studied so far can be used synchronously to reprove the fundamental theorem of algebra; this will be a guided sequence of exercises which will appear in the homework.

THEOREM 3.2.1. *Let*

$$\mathrm{P}(z) = a_n z^n + \cdots + a_1 z + a_0 \qquad n \geqslant 1, a_i \in \mathbb{C}, a_n \neq 0.$$

Then there exists z_0 *such that* $\mathrm{P}(z_0) = 0$*, i.e.,* P *admits a solution.*

In fact we will prove that the smooth function $\mathrm{P} : \mathbb{C} \to \mathbb{C}$ is surjective, which implies Theorem 3.2.1. Recall that we have the stereographic projection

$$\varphi_{\mathrm{N}} : \mathrm{S}^2 \smallsetminus \{\mathrm{N}\} \to \mathbb{R}^2$$

which is a diffeomorphism onto its image. Let us define

$$f : \mathrm{S}^2 \to \mathrm{S}^2 by$$

$$f(x) = \begin{cases} \varphi_{\mathrm{N}} \circ \mathrm{P} \circ \varphi_{\mathrm{N}}^{-1} & x \in \mathrm{S}^2 \smallsetminus \{\mathrm{N}\} \\ \mathrm{N} & x = \mathrm{N}. \end{cases}$$

LEMMA 3.2.2. *The map f, as defined above is smooth. Furthermore, f is smooth if and only if* P *is surjective.*

LEMMA 3.2.3. *The map f only has finitely many critical points. Therefore,* $\mathrm{RegVal}(f)$ *is connected.*

Now, any locally constant function on a connected space is constant. Hence, an appeal to Proposition 3.1.4 deduces that:

LEMMA 3.2.4. *The function* $\mathrm{RegVal}(f) \subset \mathrm{S}^2 \to \mathbb{N}$ *given by* $x \mapsto |f^{-1}(x)|$ *is, in fact, constant.*

LEMMA 3.2.5. *The function f takes on infinitely many different values.*

PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA. Assume that P has no solution. Then f is not surjective by Lemma 3.2.2 whence there exists some $y \in \mathrm{S}^2$ such that $f^{-1}(y)$ is empty. By definition, y is a regular value. So by Lemma 3.2.4, all other fibres over regular values are empty. We then deduce that f has nonempty fibres only at critical values. But now, by Lemma 3.2.3 we know that f only has finitely many critical points and hence only has finitely many critical values. But this contradicts Lemma 3.2.5. □

4. Application: manifolds with boundary and Brouwer's fixed point theorem

Let D^{n+1} be the $n+1$-disk:

$$\mathrm{D}^{n+1} = \{(x_0, \cdots, x_n) : \sum_{i=0}^{n} x_i^2 \leqslant 1\} \subset \mathbb{R}^{n+1}.$$

The spheres that we have seen in class are subspaces of D^{n+1}: $\mathrm{S}^n \subset \mathrm{D}^{n+1}$. We are, by now, quite close to proving one of the most famous theorems in topology.

THEOREM 4.0.1 (Brouwer). *For any $n \geqslant 1$, any continuous function $f : \mathrm{D}^n \to \mathrm{D}^n$ has a fixed point.*

4.1. Proof of Theorem 4.0.1. We list the ideas needed to prove Theorem 4.0.1 and subsequently introduce them:

(1) we need to contemplate D^n as a manifold: this requires us to introduce the notion of a *manifold with boundary*;
(2) we can classify all 1-manifolds;
(3) there is an ample supply of regular values for a smooth map f — what guarantees this is *Sard's theorem*;
(4) we can approximate any continuous function as close as we want with smooth functions.

Each of these items are quite substantial; we will only be able to make partial sketches. First, note that manifolds are modelled by open subsets of Euclidean space; in particular something that looks like

$$[0,\infty)\times\mathbb{R}^{n-1}\subset\mathbb{R}^n$$

is not a manifold. The theory of manifolds with boundary allows us to use these as local models.

Definition 4.1.1. We write the **n-dimensional Euclidean half space** as

$$\mathbb{H}^n := \{(x_1,\cdots,x_{n-1},x) : x\geqslant 0\}\subset\mathbb{R}^n$$

and define its **boundary** to be

$$\partial\mathbb{H}^n=\mathbb{R}^{n-1}\times 0\subset\mathbb{H}^n.$$

The formalism of manifolds with boundary is very similar to what we have seen so far.

Definition 4.1.2. Let X be a space. A **chart with boundary** on X is a triple $(\mathrm{U},\mathrm{V},\varphi)$ where $\mathrm{U}\subset\mathbb{H}^n$ is an open, $\mathrm{V}\subset\mathrm{X}$ is an open and $\varphi:\mathrm{U}\to\mathrm{V}$ is a homeomorphism. An **n-dimensional atlas with boundary** for X is a collection of charts with boundary $\{(\mathrm{U}_\alpha,\mathrm{V}_\alpha,\varphi_\alpha)\}$ such that $\bigcup_\alpha \mathrm{V}_\alpha=\mathrm{X}$. An atlas with boundary is said to be **smooth** if for any indices α,β the map

$$\psi_{\alpha\beta}:=\varphi_\beta^{-1}\varphi_\alpha:\varphi_\alpha^{-1}(\mathrm{V}_\alpha\cap\mathrm{V}_\beta)\to\varphi_\beta^{-1}(\mathrm{V}_\alpha\cap\mathrm{V}_\beta)$$

is a smooth.

Similar to the usual situation, we can define what it means for two smooth n-dimensional atlases with boundary to be compatible and every such is contained in a maximal one.

Definition 4.1.3. A **n-manifold with boundary** is a Hausdorff, second countable topological space M with a maximal n-dimensional smooth atlas with boundary. The **boundary** of M, denoted by $\partial\mathrm{M}$ are those points which are the images of $\{0\}\times\mathbb{R}^{n-1}=\partial\mathbb{H}^n$ under a chart. A point $x\in\mathrm{M}$ is a **interior point** if $x\in\mathrm{M}\smallsetminus\partial\mathrm{M}$. We define the **interior manifold** as $\mathrm{M}\smallsetminus\partial\mathrm{M}$ and it will be denoted by M°

We note that $\partial\mathrm{M}$ itself is a manifold which is of dimension $\dim(\mathrm{M})-1$ and M° is a smooth manifold of dimension $\dim(\mathrm{M})$. We take the convention that if we say M is a manifold, then its boundary is empty.

Example 4.1.4. Here are a list of examples of common manifolds with boundary:

(1) if M is an n-manifold, then it is also a manifold with boundary but $\partial\mathrm{M}=\emptyset$.
(2) If M is an $n-1$-manifold, then $\mathrm{M}\times[0,1]$ is a n-manifold with boundary.
(3) The disk D^n is a n-manifold with boundary and $\partial\mathrm{D}^n=\mathrm{S}^{n-1}$.

The next result extends Theorem 3.0.3 to the case of manifolds with boundary. If M is a manifold with boundary and N is a manifold, then a continuous function $f:\mathrm{M}\to\mathrm{N}$ is smooth if $f|_{\partial\mathrm{M}}$ and $f|_{\mathrm{M}\smallsetminus\partial\mathrm{M}}$ are smooth. Hence it makes sense to ask for an element $y\in\mathrm{N}$ to be a regular value for both f and f at the boundary.

Theorem 4.1.5. *Let $f:\mathrm{M}\to\mathrm{N}$ be a smooth function and $y\in\mathrm{N}$ be a regular value for both f and f at the boundary. Then $f^{-1}(y)$ is an $(m-n)$-dimensional manifold with boundary. Furthermore, $\partial(f^{-1}(y))=f^{-1}(y)\cap\partial\mathrm{M}$.*

THEOREM 4.1.6 (Hirsch's lemma). *Let* M *be compact. There is no smooth map* $f : \mathrm{M} \to \partial\mathrm{M}$ *that leaves* $\partial\mathrm{M}$ *pointwise fixed (that is:* $f(x) = x$ *for any* $x \in \partial\mathrm{M}$*).*

PROOF. The following two facts are needed to prove Hirsch's lemma:

(1) there exists at least one regular value of f;
(2) the only compact 1-manifolds with boundary are diffeomorphic to a finite disjoint union of closed intervals

Let $y \in \partial\mathrm{M}$ be a regular value for f. Then, by Theorem 1.0.8, we have that $f^{-1}(y)$ is a submanifold of M with boundary, such that

$$\partial(f^{-1}(y)) = f^{-1}(y) \cap \partial\mathrm{M}.$$

and furthermore $f^{-1}(y)$ is one-dimensional since the difference in dimension between M and its boundary is exactly one. Now, M is compact, hence $f^{-1}(y)$ must be compact as well. Therefore, by (2) $f^{-1}(y)$ is a finite disjoint union of intervals, hence its boundary, $\partial f^{-1}(z)$ has an even number of points. But this contradicts the assumption that f leaves $\partial\mathrm{M}$ pointwise fixed: indeed $\partial(f^{-1}(z)) \cong (f|_{\partial\mathrm{M}})^{-1}(z) = \{z\}$ where the last equality comes from the assumption that f leaves $\partial\mathrm{M}$ pointwise fixed.

□

THEOREM 4.1.7 (Smooth Brouwer's fixed point theorem). *In Theorem 4.0.1, we assume that* f *is a smooth map between manifolds with boundary. Then the conclusion holds.*

PROOF. Assume that f has no fixed point. We will construct a smooth map $g : \mathrm{D}^n \to \partial\mathrm{D}^n$ which leaves the boundary pointwise fixed. Let $x \in \mathrm{D}^n$; then $f(x) \neq x$ by assumption, whence determines a line L_x; we can choose a direction on L_x defined by "$f(x)$ to x." Set $g(x)$ to be the intersection of L_x with the boundary "in the prescribed direction." If you are confused, see the equation for g below. By construction g leaves $\partial\mathrm{D}^n$ pointwise fixed. We claim that g is smooth.

We calculate g explicitly. Indeed it can be written as

$$g(x) = x + t(x)v(x)$$

where $v(x)$ is the unit vector pointing from $f(x)$ to x, i.e.,

$$v(x) = \tfrac{x-f(x)}{|x-f(x)|}$$

and $t(x)$ is some positive real number, which depends on x. Since f is smooth, $v(x)$ is smooth. It suffices to prove that $t(x)$ is smooth. Since $g(x)$ is a vector on the boundary, we know that $|g(x)|^2 = 1$, hence writing $t(x)$ as t (thought of as a scalar) and $v(x)$ as v (thought of a a vector) we have

$$1 = |g(x)|^2 = (x + tv)(x + tv) = x \cdot x + 2tx \cdot v + t^2 v \cdot v.$$

Therefore

$$0 = (v \cdot v)t^2 + (2x \cdot v)t + x \cdot x - 1.$$

Knowing that v is a unit vector we further have that $t^2 + (2x \cdot v)t + x \cdot x - 1 = 0$. Now we use the quadratic equation to solve for the positive solution of t (since t must be positive!). This expresses t as a scalar multiplication and square roots, both of which are smooth functions. □

In the course of Theorem 4.1.7, we have used the classification of compact 1-manifolds.

THEOREM 4.1.8 (Classification of compact 1-manifolds). *Let* M *be a connected and compact* 1*-dimensional manifold. Then either:*

(1) M *is empty (this counts!);*
(2) if M *has no boundary, then it is diffeomorphic to* S^1*;*
(3) if M *has boundary, then it is diffeomorphic to a closed interval* $[a, b]$.

PROOF. If M is nonempty it has a point. First, assume that it has no boundary. Then each point then has a neighborhood, diffeomorphic to $(-1,1)$ (any open, connected subset of $\mathbb{R}$ must be diffeomorphic to such). Since it is compact it has finitely many such neighborhoods $\mathrm{U}_1,\cdots,\mathrm{U}_n$. Let us first assume that $n=1$. If this so, then M is diffeomorphic to $(-1,1)$ in which case it must be non-compact. Hence $n \geqslant 2$. If $n=2$ then U_1 and U_2 must intersect, otherwise they constitute a separation of M. Then there are two cases: either their union is a open interval (in which case M is again diffeomorphic to a non-compact manifold so this cannot happen) or their union forms a circle in which case we are done. Proceeding inductively and knowing that n must be finite (by compactness) we conclude that the union of these opens must always be a circle.

Now assume that M has boundary. Then each point then has a neighborhood, diffeomorphic to $(-1,1]$. The same argument then follows but instead of concluding that something is diffeomorphic to a circle, it will be the (non-disjoint) union of $(-1,1]$ and something diffeomorphic to $[-1,1)$.

□

To finish off the proof, we will use the following approximation lemma from analysis.

LEMMA 4.1.9. *Let $f:\mathrm{D}^n\to\mathrm{D}^n$ be continuous, then for every $\epsilon>0$ there exists a smooth map $g:\mathrm{D}^n\to\mathrm{D}^n$ such that $|f(x)-g(x)|<\epsilon$.*

PROOF OF BROUWER'S FIXED POINT THEOREM. Assume for contradiction that $f(x)\neq x$. Then we will contradict the smooth Brouwer's fixed point theorem Theorem 4.1.7. To do so, construct a continuous map

$$\mathrm{D}^n\to\mathbb{R}\qquad x\mapsto|\mathrm{F}(x)-x|.$$

Since D^n is compact, the extreme value theorem says that the above map admits a minimum ϵ. Since $f(x)\neq x$, we conclude that $\epsilon>0$.

For $\epsilon>0$, Lemma 4.1.9 lets us find a map $g:\mathrm{D}^n\to\mathrm{D}^n$ such that $|f(x)-g(x)|<\epsilon$. Apply the triangle inequality to get

$$|f(x)-g(x)|+|g(x)-x|\geqslant|f(x)-g(x)+g(x)-x|\geqslant\epsilon.$$

But this means that $|g(x)-x|>0$ which means that g has no fixed points! Contradicting the smooth Brouwer's fixed point theorem. □

4.2. Sard's theorem and the idea of genericity. Lemma 4.1.9 is a consequence of the **Stone-Weierstrass** theorem.

THEOREM 4.2.1. *Let $\mathrm{U}\subset\mathbb{R}^n$ be open, $\mathrm{K}\subset\mathrm{U}$ a compact set. Then for all $\epsilon>0$ and all continuous maps $f:\mathrm{U}\to\mathbb{R}$ then there exists $g:\mathrm{U}\to\mathbb{R}$ which is smooth and $|f(x)-g(x)|<\epsilon$ for all $x\in\mathrm{K}$.*

It says that the subset of smooth functions is "generic" in continuous ones. One way to formulate this is to topologize the set of continuous maps $\mathrm{Maps}_{\mathrm{cts}}(\mathrm{U},\mathbb{R})$ via the topology of uniform convergence and then say that the subset of smooth functions is dense in this space. This is the basic idea behind genericity in mathematics. As a reminder of terminology from Definition 3.0.1:

DEFINITION 4.2.2. Let $f:\mathrm{M}\to\mathrm{N}$ be a smooth map of manifolds. We say that:

(1) $x\in\mathrm{M}$ is a **regular point** of f if $\mathrm{D}f_x$ is surjective;
(2) $y\in\mathrm{N}$ is a **regular value** if any $x\in f^{-1}(y)$ is a regular point;
(3) x is a **critical point** if $\mathrm{D}f_x$ is not surjective and $y=f(x)$ is a **critical value** in this case.

Sard's theorem, which has been used already in this course, says:

THEOREM 4.2.3. *Let $f:\mathrm{M}\to\mathrm{N}$ be a smooth map. Then the set of regular values of f is a dense subset of Y. In other words, every open $\mathrm{U}\subset\mathrm{Y}$ contains a regular value.*

A full proof of Sard's theorem is somewhat outside the scope of this class as it does require new analytical ideas. However, let us discuss why one might think that Sard's theorem could be true and how one could try to approach it. First, zooming out, Sard's theorem says that regular values abound. Since critical values are the complements of regular values, we would like to prove that there aren't many critical values. We had seen an instance of this when we proved the fundamental theorem of algebra — we relied on the fact that the derivative of a polynomial map is a polynomial map and hence only has at most finitely many zero's. In fact, this shows that critical *points* are finite and hence critical values, which are bounded by critical points, are finite.

Inspired by this proof, one might try to prove that critical points are small. We are no longer contemplating polynomial maps so we have to think harder. One way in which a set can be "small" is if it is a compact subset. So maybe we can try to prove that. The mechanism by which a subset in Euclidean space could be compact is if it's closed and bounded (Heine-Borel) and so we prove:

LEMMA 4.2.4. *Let $f : \mathrm{M} \to \mathrm{N}$ be a smooth map. Then the set of critical points of f is closed.*

PROOF. A point $x \in \mathrm{M}$ is critical if and only if $\mathrm{D}f_x$ is not of full rank. Recall that a matrix has full rank if and only if each $r \times r$-minor is invertible. We think of $\mathrm{D}f_x$ as a collection of matrices that varies along x and consider any $r \times r$-minor which we call $\mathrm{M}_r f_x$. Denote this function by

$$\mathrm{M}_r f : \mathrm{M} \to \mathbb{R}^{r^2};$$

essentially this is just the matrix of partial derivatives of some chosen $r \times r$-minor of $\mathrm{D}f_x$; it is thus a continuous map. Now, the points in M such that $\mathrm{M}_r f$ is invertible is exactly the points where $\det(\mathrm{M}_r f) \neq 0$. In other words, it is the complement of $(\det(\mathrm{M}_r f)^{-1}(0))$. In other words, it is open.

Since there are only finitely many minors in a matrix, we can write the collection of regular points of f as a finite union of open sets and hence the set of critical points of f is closed. $\square$

This already lets us prove a decent theorem:

COROLLARY 4.2.5. *If* M *is compact, then the set of critical values of f is compact.*

This is quite good already: if N was not compact, then we can produce at least one regular value! Since the critical values cannot be all of N. But this is quite a weird hypothesis. In fact, in the general situation, it is quite hard to even produce a single regular value. There are a few general mechanisms by which one can produce nonempty subsets of Euclidean space and one of them comes under the name of Baire category theorems/Baire spaces. Recall the following characterization of compact subsets in general topology: a subset $\mathrm{A} \subset \mathrm{X}$ has empty interior if and only if $\mathrm{X} \smallsetminus \mathrm{A}$ is dense. The condition of empty interior is a "narrowness" condition for subsets: for example $\mathbb{Q}$ has empty interior (despite being dense) in $\mathbb{R}$.

DEFINITION 4.2.6 (Baire spaces). A space is **Baire** if any countable collection of closed sets of X with empty interior has a union with empty interior.

The point of Baire spaces is that the union or "narrow" subsets remain "narrow" and thus produces a way to construct nonempty subsets via taking complement of the union of subsets with empty interior.

LEMMA 4.2.7. *$\mathbb{R}^n$ is a Baire space: in fact so is any complete metric space is a Baire space. Any manifold is a Baire space: in fact so is any locally compact, Hausdorff space.*

REMARK 4.2.8. Actually manifolds are also complete metric spaces. We will be able to prove this after we have proved the Whitney embedding theorem.

Theorem 4.2.3 follows from the following analytic result.

THEOREM 4.2.9 (Euclidean version of Sard's theorem). *Let* $f : \mathbb{R}^n \to \mathbb{R}^m$ *be a smooth map. Then the set of regular values of* f *is dense in* $\mathbb{R}^m$*.*

Let us prove Theorem 4.2.9 where the idea of Baire-ness naturally occur.

PROOF THAT THEOREM 4.2.9 IMPLIES THEOREM 4.2.3. First, let us assume that $\mathrm{N} = \mathbb{R}^n$.

Let $x \in \mathrm{M}$ and choose a chart $(\mathrm{U}, \mathrm{V}, \varphi)$ such that $\mathrm{U} \cong \mathbb{R}^n$. Consider the critical values of $f \circ \varphi : \mathbb{R}^m \xrightarrow{\varphi} \mathrm{M} \to \mathbb{R}^n$. Then Theorem 4.2.9 tells us that the critical values of this smooth function has empty interior since its complement is dense.

On the other hand, by Lemma 4.2.4, the set of critical points of f is closed in M. We write K to be the intersection of the set of critical points of f with $\varphi(\mathrm{D}^m)$. Since φ is a homeomorphism and D^m is compact, $\varphi(\mathrm{D}^m)$ is compact. We thus conclude that K is compact and hence $f(\mathrm{K})$ is compact and has empty interior by the previous paragraph.

We now claim that $\mathbb{R}^n \smallsetminus \mathrm{RegVal}(f)$ has empty interior, which will be enough to show that $\mathrm{RegVal}(f)$ is dense. Indeed, using second countability, M can be covered by countably many subsets of the form $\varphi(\mathrm{D}^m)$ where $(\mathrm{U}, \mathrm{V}, \varphi)$ is a chart as in the second paragraph. Therefore, by the previous paragraph $\mathrm{N} \smallsetminus \mathrm{RegVal}(f)$ can be covered by countably many subsets which are compact with empty interior. By Lemma 4.2.7 below, this means that it has an empty interior and we are done.

To prove the general case: find a countable atlas $(\mathrm{U}_i, \mathrm{V}_i, \psi_i)$ for Y such that $\bigcup_i \psi_i(\mathrm{D}^n)$ covers Y. Now, the intersection of the critical values of f with $\psi_i(\mathrm{D}^n)$ is a compact subset with empty interior by the previous paragraph. Therefore, we can write $\mathrm{N} \smallsetminus \mathrm{RegVal}(f)$ again as a countable union of compact sets with empty interior and we again appeal to Lemma 4.2.7 $\square$

Sard's theorem has the following enhancement. We say that an arbitrary set A in $\mathbb{R}^n$ has **measure zero** if for every $\epsilon > 0$, there exists a countable collection of rectangular solids $\mathrm{S}_1, \mathrm{S}_2, \cdots$ such that $\mathrm{A} \subset \bigcup \mathrm{S}_i$ and.

$$\sum_{i=1}^{\infty} \mathrm{vol}(\mathrm{S}_i) < \epsilon.$$

We say that a subset $\mathrm{A} \subset \mathrm{M}$ in a manifold has measure zero if for every chart $(\mathrm{U}, \mathrm{V}, \varphi)$, the set $\varphi^{-1}(\mathrm{V} \cap \mathrm{A})$ has measure zero in $\mathbb{R}^n$.

THEOREM 4.2.10. *Let* $f : \mathrm{M} \to \mathrm{N}$ *be a smooth map of manifolds where* $\dim(\mathrm{N}) = n$*. Let* $\mathrm{C}_f := \{x \in \mathrm{M} : \mathrm{Rank}(\mathrm{D}f_x) < n\}$*, then* $f(\mathrm{C}_f)$ *has measure zero.*

This result is stronger than the previous one because any set of measure zero must have empty interior. But the converse is not true: $\mathbb{Q}$ has measure zero and so $\mathbb{R} \smallsetminus \mathbb{Q}$, the irrationals, is not measure zero but it has empty interior. However, we shall not need this result.

CHAPTER 5

The Whitney embedding theorem

The next big result that we want to prove in the theory of manifolds is Whitney's embedding theorem. Whitney's result reads as follows:

THEOREM 0.0.1 (Whitney). *Let* M *be an m-manifold with or without boundary. Then* M *admits an embedding* $\iota : \mathrm{M} \hookrightarrow \mathbb{R}^{2m+1}$.

Embeddings mean that the smooth structure on M is inherited from $\mathbb{R}^{2m+1}$ via the map ι. Recall that we had defined a manifold as an abstract topological space equipped with certain local structures. Theorem 0.0.1 states that one can make a manifold more concrete by saying that it is a certain subset of Euclidean space equipped with certain properties. We could also have taken the latter as the definition of a manifold, but the embedding that Whitney produces is not intrinsic to the manifold. Nonetheless, it is an interesting question to find the minimal number k such that M admits an embedding to $\mathbb{R}^k$; Theorem 0.0.1 is but an upper bound.

1. Embeddings and submanifolds

We now discuss the notion of an embedding and a submanifold. Before we proceed, let us add an addendum to the global submersion theorem. Given $f : \mathrm{M} \to \mathrm{N}$ satisfying the hypotheses of Theorem 3.0.3 and y a regular value of f. We have that that $f^{-1}(y)$ is a manifold of dimension $m - n$. There are two aspects of this result that we should point out. Firstly, $f^{-1}(y) \subset \mathrm{M}$ is evidently a subspace. But it enjoys a couple more crucial properties:

(1) $f^{-1}(y)$ is itself a manifold and, in a sense to be made precise soon, inherits the manifold structure from M.
(2) the induced map on tangent spaces $\mathrm{T}_x f^{-1}(y) \to \mathrm{T}_{\iota(x)}\mathrm{M}$ is injective. In fact, $\mathrm{T}_x f^{-1}(y)$ is calculated as the kernel of the map $\mathrm{T}_x\mathrm{M} \to \mathrm{T}_{f(x)}\mathrm{N}$.

It is useful to decouple both properties.

DEFINITION 1.0.1. A smooth map $f : \mathrm{Z} \to \mathrm{M}$ is said to be a **immersion at** $z \in \mathrm{Z}$ if the induced map $\mathrm{T}_z\mathrm{Z} \to \mathrm{T}_{f(z)}\mathrm{M}$ is injective. It is said to be an **immersion** if it is an immersion at all points of z.

This axiomatizes the property that we have an injection on tangent spaces. However, we warn the reader that an immersion need not be injective: we can embed $\mathbb{R}$ into $\mathbb{R}^2$ such that it crosses over at a point and it is still an immersion. Somewhat better is an injective immersion: that is, an immersion whose map of sets is injective. This notion, however, can still be quite pathological.

EXAMPLE 1.0.2 (Foliated tori). We give an interesting example of an injective immersion which is not an embedding. Consider $\mathbb{T}^2$ as $\mathrm{S}^1 \times \mathrm{S}^1$ living inside $\mathbb{C}^2$. Let $y \in (0, 1)$ be an irrational number and define

$$\gamma : \mathbb{R} \to \mathbb{T}^2 \qquad t \mapsto (e^{2\pi i t}, e^{2\pi i y t}).$$

It is clearly a smooth map; it is not hard to prove that this map is an immersion. The map above factors as

$$\mathbb{R} \xrightarrow{h: t \mapsto (e^{2\pi i t}, e^{2\pi i y t})} \mathbb{R}^2 \xrightarrow{q} \mathbb{R}^2/\mathbb{Z}^2$$

where the second map is the quotient map. We get induced maps on tangent spaces

$$\mathrm{T}_x\mathbb{R} \xrightarrow{\mathrm{D}h_x} \mathrm{T}_{h(x)}\mathbb{R}^2 \to \mathrm{D}q_{h(x)}\mathrm{T}_{\gamma(x)}\mathbb{T}^2.$$

The second map is an isomorphism because $\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ is a local diffeomorphism. On the other hand, $\mathrm{D}h_x$ is injective by looking at the derivatives. Hence the composite derivative is injective.

However, we claim that $\gamma(0)$ is a limit point of $\gamma(\mathbb{Z})$; this proves that γ cannot be a homeomorphism as $\mathbb{Z}$ does not have a limit point in $\mathbb{R}$. Let $\epsilon > 0$. Then Dirichlet's approximation theorem tells us that there exists n and m, integers such that $|yn - m| < \epsilon$. Noting that $|e^{it_1} - e^{it_2}| \leqslant |t_1 - t_2|$ for any real numbers t_1 and t_2 we get that $|e^{2\pi iyn} - 1| = |e^{2\pi iyn} - e^{2\pi im}| \leqslant |2\pi(yn - m)| \leqslant 2\pi\epsilon$. Therefore,

$$|\gamma(n) - \gamma(0)| \leqslant |(e^{2\pi n}, e^{2\pi yn}) - (1,1)| = |(1, e^{2\pi yn}) - (1,1)| < 2\pi\epsilon.$$

The problem is that γ is not a it is not a homeomorphism onto its image. Asking for this leads to the correct notion of being an "injection" of manifolds.

Definition 1.0.3. An **embedding** is an injective immersion which is a homeomorphism onto its image.

Example 1.0.4 (Embedding a circle in a torus). This example should be contrasted with Example 1.0.2. Let m, n be coprime integers. Then consider the map

$$\mathbb{R}/\mathbb{Z} \cong \mathrm{S}^1 \to \mathbb{R}^2/\mathbb{Z}^2 \cong \mathbb{T} \qquad [t] \mapsto [(mt, nt)].$$

Then this is an example of an embedding of S^1 into the torus.

An embedding is the analog, in the world of smooth manifolds, of the notion of a subspace in, in the world of topology. This coincides with a more "evident" notion of what it means to be a submanifold.

Definition 1.0.5. If M is a manifold, then a **submanifold (of dimension** n**)** is a subset $\mathrm{X} \subset \mathrm{M}$ such that there for each $x \in \mathrm{X}$ is a chart $(\mathrm{U}, \mathrm{V}.\varphi)$ of M such that $\varphi^{-1}(\mathrm{X}\cap\mathrm{V}) = \mathrm{U}\cap\mathbb{R}^n \subset \mathbb{R}^m$; here $\mathbb{R}^n \subset \mathbb{R}^m$ is an n-dimensional linear subspace.

We remark that a submanifold is itself a manifold. As a space, it inherits the topology from M as a subspace; since second countability and Hausdorfness are stable under taking subspaces it remains so. If $(\mathrm{U}, \mathrm{V}, \varphi)$ is a chart on M then

$$\mathrm{U} \cap \mathbb{R}^n \xrightarrow{\varphi|_{\mathrm{U}\cap\mathbb{R}^n}} \mathrm{X} \cap \mathrm{V}$$

defines a chart on X. To prove that they are same concept we invoke the last analysis theorem that we will use.

Lemma 1.0.6 (Immersion theorem). *Let* $\mathrm{U}_0 \subset \mathbb{R}^m$, $x \in \mathrm{U}_0$ *and suppose that we have a smooth map* $f : \mathrm{U}_0 \to \mathbb{R}^n$ *such that* $\mathrm{D}f_x$ *is injective. Then* $m \leqslant n$ *and there exists an open subset* $\mathrm{U} \subset \mathrm{U}_0$ *containing* x *and an open subset* $\mathrm{V} \subset \mathbb{R}^n$ *containing* $f(x)$ *and diffeomorphisms* $\varphi : \mathbb{R}^m \xrightarrow{\cong} \mathrm{U}$, $\psi : \mathbb{R}^n \xrightarrow{\cong} \mathrm{V}$ *such that*

(1) $\varphi(0) = x$
(2) $\psi(0) = f(x)$
(3) we have a commutative diagram

$$\begin{array}{ccc} \mathbb{R}^m & \longrightarrow & \mathrm{U} \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle f} \\ \mathbb{R}^n & \longrightarrow & \mathrm{V}, \end{array}$$

where $i(x_1, \cdots, x_m) = (x_1, \cdots, x_m, 0, \cdots 0)$.

Just like the submersion theorem expresses a submersion at x as locally a projection, the immersion theorem expresses a map with an injective derivative as locally a linear embedding.

This result gives

LEMMA 1.0.7 (Embeddings versus submanifolds). *A subset* $\mathrm{X} \subset \mathrm{M}$ *is a submanifold if and only if it is the image of an embedding.*

PROOF. Let $i : \mathrm{Y} \to \mathrm{M}$ be an embedding. We claim that $i(\mathrm{Y})$ is a submanifold. Take a chart $(\mathrm{U}, \mathrm{V}, \varphi)$ around y which maps, under i, to a chart $(\mathrm{U}', \mathrm{V}', \varphi')$ around $i(y)$. Then, applying Lemma 1.0.6 we get, up to shrinking the two charts further, a diagram

$$\begin{array}{ccc} \mathrm{U} \subset \mathbb{R}^n & \xrightarrow{\varphi} & \mathrm{Y} \\ \downarrow{\scriptstyle i_n} & & \downarrow{\scriptstyle i} \\ \mathrm{U}' \subset \mathbb{R}^m & \xrightarrow{\varphi} & \mathrm{M}. \end{array}$$

where i_n is inclusion of the first n-coordinates. Therefore, $\varphi'^{-1}(i(\mathrm{Y}) \cap \mathrm{V}') = \mathrm{U}' \cap (i_n(\mathrm{U}))$, witnessing $i(\mathrm{Y})$ as a submanifold of M.

On the other hand, let $\mathrm{X} \subset \mathrm{M}$ be a submanifold and write $i : \mathrm{X} \to \mathrm{M}$ to be the inclusion map. It suffices to prove that this is an embedding. Evidently, it is a homeomorphism onto its image. The fact that it is an immersion can be deduced immediately from the fact that if $\mathrm{U} \subset \mathbb{R}^m$ is subspace and $\mathbb{R}^n \hookrightarrow \mathbb{R}^m$ is a linear embedding, then the derivative of the inclusion $\mathbb{R}^n \cap \mathrm{U} \to \mathrm{U}$ is itself (being linear) and is therefore injective.

□

With this language, we can review the submersion theorem as:

THEOREM 1.0.8 (Addended global submersion theorem). *Let* $f : \mathrm{M} \to \mathrm{N}$ *be a smooth map and* $y \in \mathrm{N}$ *be a regular value. Then* $f^{-1}(y)$ *is a submanifold of* M *such that* $\mathrm{T}f^{-1}(y)$ *is the kernel of* $\mathrm{D}f : \mathrm{TM}|_{\mathrm{Z}} \to \mathrm{TN}$.

REMARK 1.0.9 (Normal bundles). Let $f : \mathrm{M} \hookrightarrow \mathrm{N}$ be an embedding. Then for each point $x \in \mathrm{M}$, the map $\mathrm{D}f_x : \mathrm{T}_x\mathrm{M} \to \mathrm{T}_{f(x)}\mathrm{N}$ is a linear subspace inclusion. The complement, which we denote by $\mathrm{N}_x\mathrm{M} \subset \mathrm{T}_{f(x)}\mathrm{N}$ is called the **normal space** at x of the embedding f. One can show that the conormal spaces assembles into the **normal bundle**, N_f which is a vector bundle on M of dimension $\dim(\mathrm{N}) - \dim(\mathrm{M})$.

2. The Whitney embedding theorem

Let M be a smooth manifold which, for simplicity, is assumed to be boundaryless. For the results that we will prove, we will also assume that M is compact for simplicity and to illustrate the key ideas. We want to be able to embed M into a $\mathbb{R}^{\mathrm{N}}$ and then try to make N as small as possible. This amounts to giving global coordinates for M so that any point in M can be represented by a tuple of real numbers in a way that is consistent across charts.

The idea of the Whitney embedding theorem for compact manifolds is somewhat simple but is widely applicable throughout manifold theory and large swaths of "geometric" mathematics:

(1) first we will use *partitions of unity* in order to produce a very "coarse" embedding $\mathrm{M} \to \mathbb{R}^{\mathrm{N}}$ for N very large. Indeed, locally around $x \in \mathrm{M}$ we already have a map $\varphi_{\mathrm{V}} : \mathrm{V} \hookrightarrow \mathbb{R}^m$ because of the existence of charts. The partition of unity is just there to "bump" φ_{V} to a globally-defined map on M. This N is very large and depends on the dimension of M and the number of covers of M; because M is compact this is finite.

(2) Next, we will use the method of *general projections*. For each point $x \in \mathbb{R}^{\mathrm{N}}$ we can think of it as a vector and hence consider $\mathbb{R} = \mathbb{R}\{x\} \subset \mathbb{R}^{\mathrm{N}}$ the line spanned by x. The orthogonal complement $x^{\perp}$ is then a Euclidean space of dimension $\mathrm{N} - 1$ and we have the projection away from x map

$$\mathbb{R}^{\mathrm{N}} \to \mathbb{R}^{\mathrm{N}=1} = x^{\perp}.$$

There are many of these possible projections but Sard's theorem will guarantee that a dense subset of them will indeed have the property that

$$\mathrm{M} \to \mathbb{R}^{\mathrm{N}} \to x^{\perp}$$

remains an injective immersion.

2.1. Proof of Whitney Embedding: Step 1. Throughout this section, we shall assume the following result from point-set topology:

PROPOSITION 2.1.1. *A continuous bijection $f : \mathrm{X} \to \mathrm{Y}$ from* X*, which is compact, to* Y*, which is Hausdorff, must be a homeomorphism.*

This result gives us immediately the following.

COROLLARY 2.1.2. *Let $i : \mathrm{M} \to \mathrm{N}$ be an injective immersion between manifolds where* M *is compact. Then i is an embedding.*

Corollary 2.1.2 simplifies some steps of the proofs of the embedding theorems because we are left to only check that some map is an injective immersion.

We now begin the proof by recalling a couple of definitions. Suppose that $f : \mathrm{X} \to [0, 1]$ is a continuous function of topological spaces. We set the support of a function to be:

$$\mathrm{supp}(f) := \overline{f^{-1}((0, 1])}.$$

Hence if $x \in \mathrm{X}$ is *not* in the support, we can find a neighborhood around x such that f is zero on this neighborhood.

DEFINITION 2.1.3. Let X be a topological space and $\{\mathrm{U}_i\} = \mathfrak{U}$ is a cover, assumed to be *finite*. Then a **(finite) partition of unity subordinate to** $\mathfrak{U}$ is a collection of continuous functions

$$\psi_i : \mathrm{X} \to [0, 1]$$

(1) $\mathrm{supp}(\psi_i) \subset \mathrm{U}_i$;
(2) $\sum_{i=1}^{n} \psi_i(x) = 1$ for all $x \in \mathrm{X}$.

We say that a partition of unity is **smooth** if each ψ_i is smooth.

Urysohn's Lemma, one of the key results from point set topology, is used to prove the existence of a partition of unity on a normal space. The key point is that given $\mathrm{A} \subset \mathrm{X}$ a closed subset, U an open containing A then we can construct a continuous function $f : \mathrm{X} \to [0, 1]$ such that f is 1 on A, and its support is contained entirely in U. This, in fact, characterizes normality of a space.

For opens on Euclidean space we can be more explicit with the construction of such a function.

LEMMA 2.1.4. *Given two real numbers $a < b$, there exists a smooth function*

$$f : \mathbb{R}^n \to \mathbb{R}$$

such that f is 1 *on $\overline{\mathrm{B}_{r_1}(0)}$, and is zero on $\mathbb{R}^n \setminus \mathrm{B}_{r_2}(0)$.*

The basic idea of Lemma 2.1.5 is to start with $n = 1$ and consider the smooth function

$$g(t) = \begin{cases} e^{\frac{-1}{t}} & t > 0 \\ 0 & t \leqslant 0. \end{cases}$$

This function can be turned into a function g such that for any real numbers $a < b$, we have that $g(t) = 1$ for $t \leqslant a$ and vanishes for $t \geqslant b$ by setting

$$g'(t) = \frac{f(b-t)}{f(b-t)+f(t-a)}$$

We then set f to be $g'(|x|)$; for details see [**?**, Pages 41-42]. This can be turned to a proof of the following lemma

LEMMA 2.1.5. *Let* M *be a compact manifold. Then any atlas admits a smooth partition of unity.*

We now proceed to prove step 1 of the Whitney embedding theorem.

THEOREM 2.1.6 (Weak Whitney embedding theorem). *Let* M *be a compact manifold. Then there is an embedding* $\mathrm{M} \hookrightarrow \mathbb{R}^{\mathrm{N}}$ *for* N *large enough.*

PROOF. Let m be the dimension of M. Since M is a compact, it admits a finite chart $\{(\mathrm{U}_i, \mathrm{V}_i, \varphi_i)\}_{i=1,\cdots,n}$ such that $\bigcup_{i=1}^{n} \mathrm{V}_i = \mathrm{M}$. By Lemma 2.1.5, there exists a smooth partition of unity subordinate to V_i given by smooth functions

$$\eta_i : \mathrm{M} \to \mathbb{R} \qquad i = 1, \cdots n.$$

Let us define

$$\psi_i(x) = \begin{cases} \eta_i(x)\varphi_i^{-1}(x) & x \in \mathrm{V}_i \\ 0 & \text{else.} \end{cases}$$

We note that $\eta_i(x)$ is a scalar and $\varphi_i^{-1}(x) \in \mathbb{R}^n$ and so we should interpret the above as scalar multiplication. We first claim that ψ_i is smooth. Indeed, the support of η_i is contained inside V_i and η_i is smooth by construction. Therefore we are looking at the product of two smooth functions.

Consider

$$\Phi = (\eta_1, \psi_1, \eta_2, \psi_2, \cdots, \eta_n, \psi_n) : \mathrm{M} \to \mathbb{R}^{n(m+1)};$$

We claim that Φ is an embedding.

(Smoothness) Since Φ is component-wise smooth, it is a smooth map.

(Injectivity) Assume that $\Phi(x) = \Phi(y)$; this means that they are componentwise equal. By construction, for some j, we have that $\eta_j(x) = \eta_j(y)$ but both are not $= 0$. Since the support of η_j is contained in V_j, we deduce that x, y must be in V_j. Now, $\psi_j(x) = \psi_j(y)$ as well and we can divide by $\eta_j(x) = \eta_j(y)$ to deduce that $\varphi_i^{-1}(x) = \varphi_i^{-1}(y)$. But this must mean that $x = y$ since φ_i defines a chart and thus is injective.

(Injective differential) Let $x \in \mathrm{M}$ and let j be an index such that $\eta_j(x) \neq 0$. We consider the following factorization of the map $\varphi_j^{-1}|_{\eta_j^{-1}((0,1])} : \eta_j^{-1}((0,1]) \to \mathrm{U}_j$:

$$\begin{tikzcd}
\eta_j^{-1}((0,1]) \arrow[r, "\Phi"] \arrow[ddr] & \mathbb{R}^{n(m+1)} \arrow[d, "\mathrm{proj}"] \\
& \mathbb{R}^m \arrow[d, dashed, "\cdot\frac{1}{\eta_j}"] \\
& \mathbb{R}^m.
\end{tikzcd}$$

Here proj extracts the term corresponding to ψ_j in Φ, and the dashed arrow refers to the fact that dividing η_j is only defined whenever $\eta_j \neq 0$; but this is fine on the image of $\eta_j^{-1}((0,1])$. The composite is clearly the map φ_j^{-1}. By the chain rule, i.e, the fact that derivatives compose we get that

$$\mathrm{D}_x\varphi_j^{-1} = \mathrm{D}_{\Phi(x)}(\tfrac{1}{\eta_j} \circ \mathrm{proj}) \circ \mathrm{D}_x\Phi.$$

The map $\mathrm{D}_x\varphi_j^{-1}$ is an invertible linear transformation since φ_j is a chart and thus $\mathrm{D}_x\Phi$ is injective.

(Embedding) We use Corollary 2.1.2 to conclude that Φ is a homeomorphism onto its image.

$\square$

2.2. Proof of Whitney embedding: Step 2. To proceed, as promised, we use the method of general projections. We make the following observations. Consider the linear subspace inclusion $\mathbb{R}^{\mathrm{N}-1} \subset \mathbb{R}^{\mathrm{N}}$ where we think of $\mathbb{R}^{\mathrm{N}-1}$ to be vectors of the form $(x_1, \cdots, x_{\mathrm{N}-1}, 0)$. Then we have the linear projection

$$\pi_v : \mathbb{R}^{\mathrm{N}} \to \mathbb{R}^{\mathrm{N}-1} \qquad (x_1, \cdots, x_{\mathrm{N}-1}, x_{\mathrm{N}}) \mapsto (x_1, \cdots, x_{\mathrm{N}-1}).$$

The kernel of this map is spanned by $(0, \cdots, 0, 1)$ but really spanned by any nonzero vector $v \in \mathbb{R}^{\mathrm{N}} \setminus \mathbb{R}^{\mathrm{N}-1}$. We want to study the following question:

Question 2.2.1. How can we guarantee that the composite

$$\mathrm{M} \hookrightarrow \mathbb{R}^{\mathrm{N}} \xrightarrow{\pi_v} \mathbb{R}^{\mathrm{N}-1}$$

remains an injective immersion.

Clearly, if we just consider $\mathrm{S}^1 \subset \mathbb{R}^2 \to \mathbb{R}^1$ for any projection, this cannot be made to happen. So the thing that guarantees a positive answer to the above question must involve the dimension. To proceed we make the following observations

(1) for π_v to be injective, it is necessary and sufficient that for any $x, y \in \mathrm{M}$ which are distinct, then the vector $x - y$ (which makes sense because we have already embedded M in Euclidean space) is not parallel to v.
(2) for π_v to be an immersion, we must have that $\mathrm{Ker}((\mathrm{D}\pi_v)_x) \cap \mathrm{T}_p\mathrm{M} = \{0\}$. But since π_v is linear we have that $\pi_v = (\mathrm{D}\pi_v)_x$ and so we ask that $\mathrm{T}_p\mathrm{M}$ does not contain any vector parallel to v.

Note that both conditions involve v but, rather, the whole line spanned by v. This suggests that these conditions could be reformulated using projective space.

Lemma 2.2.2 (Smooth Bertini theorem). *Let $\mathrm{N} \geqslant 2m+2$ and $i : \mathrm{M} \hookrightarrow \mathbb{R}^{\mathrm{N}}$ is an embedding (M is not assumed to be compact here). Then there exists a hyperplane projection*

$$\pi_v : \mathbb{R}^{\mathrm{N}} \to \mathbb{R}^{\mathrm{N}-1}$$

such that

$$\mathrm{M} \xrightarrow{i} \mathbb{R}^{\mathrm{N}} \xrightarrow{\pi_v} \mathbb{R}^{\mathrm{N}-1} = v^{\perp}$$

remains an injective immersion.

Proof. Let us regard $\mathbb{RP}^{\mathrm{N}-1}$ as a quotient of $\mathbb{R}^{\mathrm{N}} \setminus 0$. Consider the maps

$$\iota : \mathrm{M} \times \mathrm{M} \setminus \Delta_{\mathrm{M}} \to \mathbb{RP}^{\mathrm{N}-1} \qquad (x, y) \mapsto [x - y]$$
$$\iota' : \mathrm{TM} \setminus \mathrm{M} \to \mathbb{RP}^{\mathrm{N}-1} \qquad (x, w) \mapsto [w].$$

By the discussions above, we remark that the following are equivalent:

(1) $(\pi_v)|_{\mathrm{M}}$ is an injective immersion;
(2) v is not in the image of either ι or ι'.

Now, the dimension of $\mathrm{M} \times \mathrm{M} \setminus \Delta_{\mathrm{M}}$ and $\mathrm{TM} \setminus \mathrm{M}$ are both $2m$-dimensional. Note that ι and ι' are smooth. Since $2m < \mathrm{N} - 1$, the exam problem (aka Sard's theorem) says that $[v]$ is disjoint from the images of ι and ι' if and only if $[v]$ is a regular value. By Sard's theorem, there is a dense set of these. □

Theorem 2.2.3 (Compact Whitney embedding). *Let M be a compact manifold of dimension m. Then it can be embedded into a $\mathbb{R}^{2m+1}$.*

Proof. By Theorem 2.1.6 we have an embedding $\mathrm{M} \hookrightarrow \mathbb{R}^{\mathrm{N}}$. If $\mathrm{N} \leqslant 2m + 1$ we are done. Otherwise, repeatedly apply Lemma 2.2.2.

□

CHAPTER 6

Transversality

CHAPTER 7

Integration on manifolds

Recall that the fundamental theorem of calculus states (whenever it applies) that integration is an anti-derivative:

$$\int_0^1 \frac{\partial f}{\partial x} dx = f(1) - f(0).$$

Stokes' theorem is a massive generalization of this result that is applicable on manifolds. This is the subject of the last part of this course.

THEOREM 0.0.1 (Kelvin-Stokes). *Let* M *be a compact, oriented* m*-dimensional manifold, possibly with boundary. Let*

$$\omega \in \Omega^{m-1}(\mathrm{M}).$$

Then we have an equality

$$\int_{\mathrm{M}} d\omega = \int_{\partial \mathrm{M}} \omega.$$

While the proper attribution of Stokes' theorem in the above form should involve Kelvin, we will stick to calling the above Stokes' theorem. Theorem 0.0.1 is one of the most beautiful and fundamental equality of mathematics and leads to an entirely new world of mathematics such as de Rham cohomology.

The first order of business is to make sense of what the characters involved in Stokes theorem are. In order:

(1) what ω is,
(2) what $d\omega$ is,
(3) what $\int$ means.

1. 1-forms

We first make sense of what ω is; it is what's called a **differential form**. This is an example of a section of a vector bundle:

DEFINITION 1.0.1. Let $p : \mathcal{E} \to \mathrm{M}$ be a smooth vector bundle on a smooth manifold M. A **smooth section** of $\mathcal{E}$ is a smooth map $s : \mathrm{M} \to \mathcal{E}$ such that $p \circ s = \mathrm{id}_{\mathrm{M}}$.

Unwinding definitions, this just means that we have a function:

$$x \in \mathrm{M} \mapsto s(x) \in \mathcal{E}$$

and we ask that this is smooth; note that $\mathcal{E}$ itself is a manifold so this request makes sense.

Now, recall that, over every point $x \in \mathrm{M}$, TM is just the vector space $\mathrm{T}_x\mathrm{M}$ the tangent space at M. We will define $\mathrm{T}^*_x\mathrm{M}$ to be

$$\mathrm{T}^*_x\mathrm{M} = (\mathrm{T}_x\mathrm{M})^\vee$$

the $\mathbb{R}$-linear dual of $\mathrm{T}_x\mathrm{M}$. By the same reasoning as in the case of TM, these assemble into a smooth vector bundle of rank $\dim(\mathrm{M})$ which we denoted by $\mathrm{T}^*\mathrm{M}$ and is called the *cotangent bundle*; it is again a manifold of dimension $2\dim(\mathrm{M})$.

DEFINITION 1.0.2. A **1-form** is a smooth section of $\mathrm{T}^*\mathrm{M}$.

Unwinding definitions: a 1-form assigns to each $x \in \mathrm{M}$, a vector $s(v) \in \mathrm{T}_x^*\mathrm{M}$ and the assignment $x \mapsto s(v)$ must be smooth. While forms are important for integration, it is useful to consider the dual notion:

DEFINITION 1.0.3. A **vector field** is a smooth section of TM.

REMARK 1.0.4 (Nowhere vanishing). An important class of sections are those which are **nowhere vanishing**: for each $x \in \mathrm{X}$, $s(x) \neq 0$.

EXAMPLE 1.0.5 (The form df). We give an example of a 1-form. Let $f : \mathrm{M} \to \mathbb{R}$ be a smooth function. Then the 1-form df is the section given at each $x \in \mathrm{M}$ by

$$df(x) : \mathrm{T}_x\mathrm{M} \to \mathbb{R} \qquad \mathrm{X} \mapsto \mathrm{X}(f).$$

Let us claim that df is smooth. Let $(\mathrm{U}, \mathrm{V}, \varphi)$ be a chart around $x \in \mathrm{M}$; given this choice we have an isomorphism

$$\mathrm{T}_x\mathrm{M} \cong \mathbb{R}\{\tfrac{\partial}{\partial x_1}, \cdots, \tfrac{\partial}{\partial x_m}\}. \tag{1.0.6}$$

Given this isomorphism we can let dx_i to be the basis dual to $\frac{\partial}{\partial x_i}$. So that

$$\mathrm{T}_x^*\mathrm{M} = \mathbb{R}\{dx_1, \cdots, dx_m\}.$$

This means that, on the open patch V, we can write a 1-form as a function

$$s(x) = \sum_{i=1}^{m} s_i(x)dx_i.$$

Unpacking the isomorphism (1.0.6) we note that, again on the open patch V,

$$df(x) = \sum_{i=1}^{m} \tfrac{\partial f}{\partial x_i}(x)dx_i.$$

Since $x \mapsto \frac{\partial f}{\partial x_i}(x)$ is smooth for each $i = 1, \cdots, n$ the form is indeed smooth.

EXAMPLE 1.0.7 (Forms on Euclidean space). As a particular example of Example 1.0.5 we consider the smooth functions given by the projections (onto the i-th coordinate): $\pi_i : \mathbb{R}^n \to \mathbb{R}$. We usually write the corresponding form here as dx_i; so with the notation of Example 1.0.5 we write $dx_i := d\pi_i$. Explicitly, this is the section

$$dx_i : \mathbb{R}^n \to \mathrm{T}^*\mathbb{R}^n$$
$$(x_1, \cdots, x_n) \mapsto dx_i \in \mathrm{T}_x^*\mathrm{M} = (\mathrm{T}_x\mathrm{M})^\vee.$$

So far there is nothing surprising.

EXAMPLE 1.0.8 (The fundamental form of the circle). Consider $\mathrm{S}^1 \subset \mathbb{R}^2$. We will make sense of the **fundamental form** on the circle which is given by the equation

$$\omega = -ydx + xdy.$$

More precisely, we have that $-ydx + xdy$ is a 1-form on $\mathbb{R}^2$ and ω is given by the restriction of this form to S^1.

First, we study a vector field on the circle. Recall that we have the map

$$\gamma : \mathbb{R} \to \mathrm{S}^1 \qquad t \mapsto (\cos t, \sin t).$$

While this is not a diffeomorphism (or even a homeomorphism) it is a submersion; the derivative is given by

$$c'(t) = (-\sin t, \cos t) = (-y, x)$$

which is never zero on the circle. We can interpret this as a vector field denoted by

$$\mathrm{X} := -y\tfrac{\partial}{\partial x} + x\tfrac{\partial}{\partial y}.$$

What this means is that if $\iota : \mathrm{S}^1 \to \mathbb{R}^2$ is the standard embedding of the circle in $\mathbb{R}^2$, then

$$(\iota_* \mathrm{X})_{(x,y)} = -y\tfrac{\partial}{\partial x} + x\tfrac{\partial}{\partial y},$$

where $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ forms the standard basis for $\mathrm{T}_{(x,y)}\mathbb{R}^2$.

We claim that there exists a 1-form, denoted by ω, such that $\omega(\mathrm{X}) = 1$ at all points of S^1. To do so, we write

$$\omega(\mathrm{X}) = adx + bdy$$

and our goal is to solve for a and b in terms of x and y. Unwinding definitions we have that

$$\omega(\mathrm{X}) = adx + bdy(-y\tfrac{\partial}{\partial x} + x\tfrac{\partial}{\partial y}) = -ay + bx.$$

We note that $x^2 + y^2 = 1$ is a constrain since we are on S^1, we conclude that $a = -y$ and $b = x$ is a solution. So we have that

$$\omega = -ydx + xdy$$

is a nowhere vanishing 1-form on the circle; this is the fundamental form on the circle and we will revisit this again later.

Lastly, we will use the following notation.

Definition 1.0.9. Let $\mathcal{E} \to \mathrm{M}$ be a smooth vector bundle. If s, s' are sections, then we can define

$$s + s' \qquad \lambda s, \lambda \in \mathbb{R}$$

by performing the operations fibrewise and noting that the resulting section is still smooth. We will denote by

$$\Gamma(\mathrm{M}, \mathcal{E}),$$

the **(vector) space of sections of** $\mathcal{E}$. The space of 1-forms is denoted by

$$\Gamma(\mathrm{M}, \mathrm{T}^*\mathrm{M}) =: \Omega^1(\mathrm{M}).$$

2. Multilinear algebra

Now that we have explored the idea of 1-forms, we want to explore the idea of k-forms for $k \geqslant 1$. In the end, whatever a k-form is, it is something that we can integration. For example, if $\mathrm{U} \subset \mathbb{R}$ is an open subset of $\mathbb{R}$ and $f : \mathrm{U} \to \mathbb{R}$ is an integrable function, then we are familiar with the expression

$$\int_{\mathrm{U}} f(x)dx.$$

We should think of the expression "$f(x)dx$" as something that one can integrate and obtain the usual answer from one-variable calculus. This then leads to the theory of integration on 1-manifolds by gluing together these values.

Similarly, if $\mathrm{U} \subset \mathbb{R}^n$ an open and $f : \mathrm{U} \to \mathbb{R}$ is integrable we should have seen the integral

$$\int_{\mathrm{U}} f(x)dx_1 \cdots dx_n.$$

We want to think of "$fdx_1 \cdots dx_n$" as an n-form on U and this is something that we can integrate and obtain the usual answer from multivariable calculus, out of which we obtain a theory of integration on n-manifolds.

2.1. Some multilinear algebra. Let us start with an example.

Example 2.1.1. Consider the polynomial ring in one variable, $\mathbb{R}[x]$. Then this object is a ring, but we can also endow it with a **grading**: this is a decomposition of the underlying vector space as

$$\mathbb{R}[x] = \bigoplus_{n \geqslant 0} \mathbb{R}\{x^n\}.$$

Concretely, this means that any polynomial $p(x)$ can be written as

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots a_1 x + a_0 \qquad a_j \in \mathbb{R}$$

where each component $a_j x^j$ is said to have (homogeneous) **degree** j. Note that multiplication in $\mathbb{R}[x]$ takes an element of degree i and an element in degree j into an element of degree $i+j$:

$$(a_j x^j) \cdot (a_i x^i) = (a_i a_j x^{i+j}).$$

Hence $\mathbb{R}[x]$ is an example of a **graded, associative $\mathbb{R}$-algebra**: an ring R, equipped with a ring homomorphism $\mathbb{R} \to \mathrm{R}$ such that R can be written as

$$\mathrm{R} = \bigoplus_{n \geqslant 0} \mathrm{R}_n$$

where each R_n is a $\mathbb{R}$-vector space, multiplication does:

$$\mathrm{R}_i \mathrm{R}_j \subset \mathrm{R}_{i+j}.$$

and $1 \in \mathrm{R}_0$. In the case of $\mathbb{R}[x]$ the multiplication happens to be commutative.

To proceed, we will introduce the following object which is somewhat sophisticated but is a very fundamental mathematical structure.

Definition 2.1.2. A **$\mathbb{R}$-linear commutative differential graded algebra** or **$\mathbb{R}$-cdga** for short is

(1) a $\mathbb{N}$-indexed collection of vector spaces R^n, $n \geqslant 0$
(2) a linear map $d : \mathrm{R}^n \to \mathrm{R}^{n+1}$, $n \geqslant 0$
(3) An graded $\mathbb{R}$-algebra structure on $\mathrm{V} := \bigoplus_{n \geqslant 0} \mathrm{V}_n$.

These are subject to:

(1) the multiplication is **graded commutative**

$$a \cdot b = (-1)^{|a||b|} b \cdot a$$

where $|x|$ indicates the degree of x (which vector space it lives in).
(2) The linear map d satisfies $d \circ d = 0$.
(3) The map d satisfies the (graded) Liebniz rule

$$d(ab) = (da)b + (-1)^{|a|} a db.$$

We denote such a thing as a pair $(\mathrm{R}^\star, d)$.

The first thing to observe about an $\mathbb{R}$-cdga is that the odd degree elements square to zero. Indeed, if $|x|$ is odd, then

$$x \cdot x = (-1)^{\mathrm{odd}} x \cdot x = -x \cdot x.$$

Therefore $x \cdot x = 0$. While the notion of $\mathbb{R}$-cdga is abstract, it succinctly encodes the structure of $\Omega^j(\mathrm{M})$ for every manifold M. Towards this end, we introduce multilinear algebra and its attendant constructions.

Construction 2.1.3 (Tensor products). Let V, W be $\mathbb{R}$-vector spaces. Its **tensor product** is defined as the quotient of the free vector space on the set $\mathrm{V} \times \mathrm{W}$[1] by the subspace spanned by the elements:

(1) $((v + v'), w) - (v, w) - (v', w)$
(2) $(v, (w + w')) - (v, w) - (v, w')$
(3) $(\lambda v, w) - \lambda(v, w)$ for $\lambda \in \mathbb{R}$
(4) $(v, \lambda w) - \lambda(v, w)$ for $\lambda \in \mathbb{R}$.

The equivalence class of (v, w) is denoted by $v \otimes w$. We denote by $\pi : \mathrm{V} \times \mathrm{W} \to \mathrm{V} \otimes \mathrm{W}$ the map that sends $(v, w) \mapsto v \otimes w$; by construction it is a bilinear map of $\mathbb{R}$-vector spaces.

The relations in the above construction are exactly the relations one would encounter when formulating bilinearity. It is designed to satisfy the following universal property.

Proposition 2.1.4 (Universal properties of the tensor product). *Let* V, W *be* $\mathbb{R}$*-vector spaces. Then any bilinear map* $\mathrm{V} \times \mathrm{W} \to \mathrm{X}$ *extends uniquely to a linear map* $\mathrm{V} \otimes \mathrm{W} \to \mathrm{X}$ *rendering the following diagram commutative*

$$\begin{tikzcd} \mathrm{V} \times \mathrm{W} \arrow[r] \arrow[d, "\pi"] & \mathrm{X} \\ \mathrm{V} \otimes \mathrm{W}. \arrow[ur, dashed, "!"'] & \end{tikzcd}$$

Remark 2.1.5. Given maps $f : \mathrm{V}_1 \to \mathrm{W}_1$ and $g : \mathrm{V}_2 \to \mathrm{W}_2$, we can construct a map $f \otimes g : \mathrm{V}_1 \otimes \mathrm{V}_2 \to \mathrm{W}_1 \otimes \mathrm{W}_2$ such that $(f \otimes g)(v_1 \otimes v_2) = f(v_1) \otimes g(v_2)$. The way to do this is to use Proposition 2.1.4: let $\mathrm{X} = \mathrm{W}_1 \otimes \mathrm{W}_2$ and consider the bilinear map

$$\mathrm{V}_1 \times \mathrm{V}_2 \to \mathrm{W}_1 \otimes \mathrm{W}_2 \qquad (v_1, v_2) \mapsto f(v_1) \otimes g(v_2).$$

Then the universal property of the tensor product produces a map $f \otimes g : \mathrm{V}_1 \otimes \mathrm{V}_2 \to \mathrm{W}_1 \otimes \mathrm{W}_2$.

The point of $\mathrm{V} \otimes \mathrm{W}$ is to convert bilinear maps to linear maps. Hence it is the central object of "bilinear algebra." The subject of multilinear algebra is simply the study of linear maps $\mathrm{V}_1 \otimes \cdots \otimes \mathrm{V}_n \to \mathrm{W}$.

Remark 2.1.6. We can reformulate the definition of $\mathbb{R}$-algebra diagramatically: R is a $\mathbb{R}$-vector space equipped with a map $\mu : \mathrm{R} \otimes \mathrm{R} \to \mathrm{R}$ such that the diagrams

$$\begin{tikzcd} \mathrm{R} \otimes \mathrm{R} \otimes \mathrm{R} \arrow[r, "\mu \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \mu"'] & \mathrm{R} \otimes \mathrm{R} \arrow[d, "\mu"] \\ \mathrm{R} \otimes \mathrm{R} \arrow[r, "\mu"] & \mathrm{R}, \end{tikzcd}$$

and

$$\begin{tikzcd} \mathbb{R} \otimes \mathrm{R} \arrow[r, "1 \otimes \mathrm{id}"] \arrow[dr, "\cong"'] & \mathrm{R} \otimes \mathrm{R} \arrow[d, "\mu"] & \mathrm{R} \otimes \mathbb{R}. \arrow[l, "\mathrm{id} \otimes 1"'] \arrow[dl, "\cong"] \\ & \mathrm{R} & \end{tikzcd},$$

commutes.

Multilinear algebra comes in a variety of flavors. For example we can ask that a bilinear map

$$h : \mathrm{V} \times \mathrm{V} \to \mathrm{W}$$

be **symmetric**:

$$h(v_1, v_2) = h(v_2, v_1).$$

[1] If X is a set, the **free vector space** on X is the vector space denoted by $\mathbb{R}\{\mathrm{X}\}$ with basis elements of X. In other words, any vector in this vector space is a finite linear combination $\sum \lambda_i x_i$ where $\lambda_i \in \mathbb{R}$ and $x_i \in \mathrm{X}$. Here, we forget that $\mathrm{V} \times \mathrm{W}$ has the structure of a $\mathbb{R}$-vector space, think about it as a set and then take its free vector space.

This is generalized by asking that a multilinear map

$$h : \mathrm{V} \times \cdots \times \mathrm{V} \to \mathrm{W}$$

be **symmetric** for any permutation $\sigma \in \Sigma_n$ we have that

$$h(v_1, \cdots, v_n) = h(v_{\sigma(1)}, \cdots, v_{\sigma(n)}).$$

In this case, any multilinear map defines a linear map $\mathrm{V} \otimes \cdots \otimes \mathrm{V} \to \mathrm{W}$ and a symmetric one factors through a further quotient called the n-th symmetrization of V:

$$\mathrm{Sym}^n(\mathrm{V}) \to \mathrm{W};$$

the vector space $\mathrm{Sym}^n(\mathrm{V})$ is a further quotient of $\mathrm{V}^{\otimes n}$ by the action of Σ_n. In differential topology what is more prominent are alternating tensors. Let us first recall that we have a group homomorphism

$$\mathrm{sgn} : \Sigma_n \to \mathbb{Z}/2 = \{\pm 1\}$$

which is determined by asking that sgn(transposition) $= -1$ and that sgn is a group homomorphism.

Definition 2.1.7. A multilinear map

$$h : \mathrm{V} \times \cdots \times \mathrm{V} \to \mathrm{W}$$

is said to be **alternating** if

$$h(v_1, \cdots, v_n) = \mathrm{sgn}(\sigma) h(v_{\sigma(1)}, \cdots, v_{\sigma(n)}).$$

The **n-th exterior power** of a $\mathbb{R}$-vector space is the quotient of $\mathrm{V}^{\otimes n}$ by the subspace spanned by elements

$$\mathrm{sgn}(\sigma)(v_1 \otimes \cdots \otimes v_n) - (v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)})$$

We will denote this vector space as $\bigwedge^n \mathrm{V}$. It comes equipped with an alternating multilinear map

$$\mathrm{V}^{\otimes n} \to \bigwedge^n \mathrm{V}.$$

We will denote the image of $v_1 \otimes \cdots \otimes v_n$ as

$$v_1 \wedge \cdots \wedge v_n.$$

An expression of the form $v_1 \wedge \cdots \wedge v_n$ is called a **pure alternating tensor**.

Remark 2.1.8 (Antisymmetry). One of the key properties of $\bigwedge^k \mathrm{V}$ is **antisymmetry**. Note that the sign of a transposition is always negative, hence we always get a relation

$$v_1 \wedge \cdots v_i \wedge v_{i+1} \wedge \cdots v_k = -v_1 \wedge \cdots v_{i+1} \wedge v_i \wedge \cdots v_k.$$

One consequence of this is the following:

$$v_1 \wedge v_1 = -v_1 \wedge v_1$$

and thus equal to zero.

Remark 2.1.9 (Basis elements). Let us examine the example of $\bigwedge^2 \mathbb{R}^n$. So $\mathrm{V} = \mathbb{R}^n$, an n-dimensional $\mathbb{R}^n$-vector space and we are taking the 2nd exterior power. Choose a basis $e_1, \cdots, e_n$ for $\mathbb{R}^n$ (say, the standard basis). Then we claim that a basis is given by

$$\{e_i \wedge e_j : 1 \leqslant i < j \leqslant n\}$$

Indeed, we start out with a basis for the tensor product $e_i \otimes e_j, i, j \in \{1, \cdots, n\}$. But the alternating relation tells us that if a basis vector appears twice then it is zero by antisymmetry, Remark 2.1.8. Furthermore, if the basis does not appear in the proper order (like $e_5 \wedge e_2$) then the alternating relation says that they differ by a negative sign (so $e_5 \wedge e_2 = -e_2 \wedge e_5$).

More generally, a basis for $\wedge^k(\mathbb{R}^n)$ is given by

$$\{e_{i_1} \wedge e_{i_2} \wedge \cdots e_{i_k} : 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n\}$$

Hence, by standard combinatorics, we have:

$$\dim \bigwedge^k(\mathbb{R}^n) = \binom{n}{k}. \tag{2.1.10}$$

REMARK 2.1.11. Not all elements of $\bigwedge^n \mathrm{V}$ are pure alternating tensors. For example, the element $e_1 \wedge e_2 + e_3 \wedge e_4 \in \bigwedge^2 \mathbb{R}^n$ cannot be written as $v_1 \wedge v_2$ for some $v_1, v_2 \in \mathbb{R}^n$.

The following universal property of exterior powers follow by construction.

PROPOSITION 2.1.12 (Universal property of exterior powers). *Given an alternating multilinear map $h : \mathrm{V}^{\times k} \to \mathrm{W}$, there exists a unique linear map $\bigwedge^k \mathrm{V} \to \mathrm{W}$ rendering the following diagram commutative*

REMARK 2.1.13. Given a linear map $f : \mathrm{V} \to \mathrm{V}'$ we get an induced map

$$\bigwedge^k(f) : \bigwedge^k(\mathrm{V}) \to \bigwedge^k(\mathrm{V}')$$

which can be described on pure alternating tensors as:

$$\bigwedge^k(f)(v_1 \wedge \cdots \wedge v_k) = f(v_1) \wedge \cdots \wedge f(v_k).$$

The formation of $\bigwedge^k(f)$ satisfies:

(1) $\bigwedge^k(f \circ g) = \bigwedge^k(f) \circ \bigwedge^k(g)$
(2) $\bigwedge^k(\mathrm{id}) = \mathrm{id}$.

EXAMPLE 2.1.14 (Top exterior power and the determinant). Let V be an n-dimensional vector space. According to (2.1.10), $\dim \bigwedge^n(\mathrm{V}) = 1$ and is spanned by the vector $e_1 \wedge e_2 \wedge \cdots e_n$. We explain how this lets us extract the determinant of matrices. Let $n = 2$ and say that we have

$$\mathrm{A} = \begin{pmatrix} a & b \\ c & d. \end{pmatrix}$$

Now, the universal property of the exterior powers Proposition 2.1.12 yields a map

$$\bigwedge^2(\mathrm{A}) : \bigwedge^2(\mathrm{V}) \to \bigwedge^2(\mathrm{V});$$

we will explain how to compute this. Now note that we are considering a $\mathbb{R}$-linear map of 1-dimensional vector space, so we are looking for a number: in other words we are looking where the basis element $e_1 \wedge e_2$ gets sent to.

We compute:

$$\begin{aligned} \mathrm{A}(e_1) \wedge \mathrm{A}(e_2) &= (ae_1 + ce_2) \wedge (be_1 + de_2) \\ &= abe_1 \wedge e_1 + ade_1 \wedge e_2 + cbe_2 \wedge e_1 + cde_2 \wedge e_2 \\ &= (ad - bc)e_1 \wedge e_2. \end{aligned}$$

Therefore this produces the determinant of a 2×2-matrix. The result, which can actually be taken as the definition of the determinant is that if

$$\mathrm{A} : \mathbb{R}^n \to \mathbb{R}^n$$

is an $n \times n$-matrix with respect to the standard basis then the induced map

$$\bigwedge^n : \bigwedge^n(\mathbb{R}^n) \to \bigwedge^n(\mathbb{R}^n)$$

sends the basis $e_1 \wedge \cdots \wedge e_n$ to $\det(\mathrm{A})e_1 \wedge \cdots \wedge e_n$. This is a definition for the determinant of a matrix that is as good as any other.

CONSTRUCTION 2.1.15. We now construct, for a vector bundle $\mathcal{E}$, its k-th exterior bundle. Let $p : \mathcal{E} \to X$ be a smooth vector bundle, its ***k*-th exterior bundle** is constructed by constructing $\bigwedge^k(\mathcal{E}_x)$ to each fibre. It is then obtained by gluing the induced map on local trivializations: if

$$\psi : p^{-1}(U) = \sqcup_{x \in U} \mathcal{E}_x \to U \times \mathbb{R}^n;$$

then the induced maps

$$\bigwedge^k(\psi) : \sqcup_{x \in U} \bigwedge^k(\mathcal{E}_x) \to U \times \bigwedge^k(\mathbb{R}^n)$$

is used to produce a vector bundle $\bigwedge^k(\mathcal{E}) \to X$ of rank $\binom{\mathrm{rank}(\mathcal{E})}{k}$.

The space of ***k*-forms** is then the $\mathbb{R}$-vector space of sections of the bundle $\bigwedge^k T^*M$; we denote this by $\Omega^k(M)$. By definition, the space of **0-forms** is the $\mathbb{R}$-vector space of smooth functions $M \to \mathbb{R}$; we denote this by $\Omega^0(M)$.

EXAMPLE 2.1.16 (Local description). Let M be a manifold and let (U, V, φ) be a chart around x. Then we can trivialize $\bigwedge^k T^*M$ around x using this chart. A k-form ω on M restricts to a k-form on V and, using the trivialization, we can write it as

$$\omega = \sum f_{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}. \tag{2.1.17}$$

for $f_{i_1 \cdots i_k} : V \to \mathbb{R}$ a smooth function defined on V.

Sections of the p-exterior bundle, for $p \geqslant 0$, turns out to be the key object of study in differential topology. In the next construction, we contemplate the wedge product of sections.

CONSTRUCTION 2.1.18 (Wedge product). Let $p, q \geqslant 0$, then we have the wedge product

$$\wedge : \Omega^p(M) \otimes \Omega^q(M) \to \Omega^{p+q}(M).$$

This is fibrewise defined as

$$\wedge : \bigwedge^p(T_x^*M) \otimes \bigwedge^q(T_x^*M) \to \bigwedge^{p+q}(T_x^*M)$$
$$\omega \otimes \omega' \mapsto \omega \wedge \omega'.$$

In other words, if $\omega : M \to \bigwedge^p T^*M, \theta : M \to \bigwedge^q T^*M$ are two sections, then $\omega \wedge \theta$ is the section that sends

$$x \mapsto \omega(x) \wedge \theta(x) \in \bigwedge^{p+q} T_x^*M.$$

It is clearly seen to be smooth (expand out ω, θ in coordinates as in (2.1.17) and then note that the functions just multiply).

By construction, $\wedge$ enjoys

(1) $(\omega + \omega') \wedge \theta = \omega \wedge \theta + \omega' \wedge \theta$;
(2) $\omega \wedge (\theta + \theta') = \omega \wedge \theta + \omega \wedge \theta'$;
(3) $(\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3)$;
(4) it has graded commutativity

$$\omega \wedge \theta = (-1)^{|\omega||\theta|} \theta \wedge \omega.$$

2.2. The exterior derivative/de Rham differential. We have now constructed the wedge product, producing a graded commutative algebra structure on $\bigoplus_{p \geqslant 0} \Omega^p(M)$. What is missing is the differential d, a linear map between adjacent gradings such that $d^2 = 0$. This is the subject of the next construction.

CONSTRUCTION 2.2.1 (Exterior derivative/de Rham differential). We construct the **exterior derivative** (also called the **de Rham differential**) which is an $\mathbb{R}$-linear map

$$d : \Omega^p(M) \to \Omega^{p+1}(M).$$

It is defined using charts. On a chart $(\mathrm{U}, \mathrm{V}, \varphi)$ a p-form can be written as

$$\omega = \sum f_{i_1 \cdots i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

Then

$$d\omega := \sum_{i=1}^{m} \sum \frac{\partial f_{i_1 \cdots i_p}}{\partial x_i} dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

Note that many terms of $d\omega$ disappear or acquires a negative. Actually extending the definition of d to an arbitrary manifold requires one to be slightly careful. We first introduce the operation of pullback.

Construction 2.2.2 (Pullback of differential forms). Let us construct the pullback of forms: we will associate to a smooth function of manifolds $f : \mathrm{M} \to \mathrm{N}$ a linear map

$$f^* : \Omega^k(\mathrm{N}) \to \Omega^k(\mathrm{M}) \qquad k \geqslant 0$$

such that

$$(fg)^* = g^* f^* \qquad (\mathrm{id})^* = \mathrm{id}.$$

First, let $k = 0$. Then we have a map

$$f^* : \mathcal{O}(\mathrm{N}) \to \mathcal{O}(\mathrm{M}) \qquad g : \mathrm{N} \to \mathbb{R} \mapsto g \circ f : \mathrm{M} \to \mathrm{N} \to \mathbb{R}.$$

For $k \geqslant 1$, we note that each point $x \in \mathrm{M}$ we have a linear map

$$\mathrm{D}f_x : \mathrm{T}_x\mathrm{M} \to \mathrm{T}_{f(x)}\mathrm{N};$$

and therefore we have a dual map

$$\mathrm{D}f_x^* := (\mathrm{D}f_x)^\vee : \mathrm{T}^*_{f(x)}\mathrm{N} \to \mathrm{T}^*_x\mathrm{M}.$$

Given a section $s : \mathrm{N} \to \mathrm{T}^*\mathrm{N}$, we can then construct a section

$$\mathrm{D}f^*(s) : \mathrm{M} \to \mathrm{T}^*\mathrm{M} \qquad x \mapsto \mathrm{D}f_x^*(s(f(x))).$$

This defines a map

$$f^* : \Omega^1(\mathrm{N}) \to \Omega^1(\mathrm{M}) \qquad s \mapsto \mathrm{D}f^*(s),$$

which is easily seen to be $\mathbb{R}$-linear map of vector spaces. In the same way, we can define a map

$$f^* : \Omega^k(\mathrm{N}) \to \Omega^k(\mathrm{M}) \qquad k \geqslant 2;$$

which, on each fibre, is induced by

$$\bigwedge^k (\mathrm{D}f_x^*) : \bigwedge^k \mathrm{T}^*_{f(x)}\mathrm{N} \to \bigwedge^k \mathrm{T}^*_x\mathrm{M}.$$

That f^* preserves the smoothness of sections will be explained in Example 2.2.4.

Tracing through the definitions, we can make the following simple calculation about pullbacks

Proposition 2.2.3. *Given a smooth map of manifolds $f : \mathrm{M} \to \mathrm{N}$, there exists a map*

$$f^* : \bigoplus_{p \geqslant 0} \Omega^p(\mathrm{N}) \to \bigoplus_{p \geqslant 0} \Omega^p(\mathrm{M})$$

such that:

(1) *f^* is, on each component, a $\mathbb{R}$-linear map*

$$f^* : \Omega^k(\mathrm{N}) \to \Omega^k(\mathrm{M}) \qquad k \geqslant 0$$

given by

(2) *f^* commutes with wedge products*

$$f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta).$$

(3) *f^* commutes with derivatives of functions:*

$$f^* d = d f^*.$$

EXAMPLE 2.2.4 (Pullback in coordinates). Let us compute f^* in coordinates, at least for 1-forms. For p-forms, the added complication is purely notational. Let $f : \mathrm{M} \to \mathrm{N}$ be a smooth map and $x \in \mathrm{M}$ such that, around x, f is expressed as a smooth map $f = (f^1, \cdots, f^n) : \mathrm{U} \to \mathrm{V}$ where $\mathrm{U} \subset \mathbb{R}^m$ with coordinates $(y_1, \cdots, y_m)$ and $\mathrm{V} \subset \mathbb{R}^n$ with coordinates $(x_1, \cdots, x_n)$. Then $\omega \in \Omega^1(\mathrm{N})$ can be written locally in U as an expression of the form

$$\omega = \sum_{i=1}^{n} g_i dx_i.$$

We then calculate

$$\begin{aligned} f^*(\omega) &= \sum_{i=1}^{n} f^*(g_i dx_i) \\ &= \sum_{i=1}^{n} (g_i \circ f) f^*(dx_i) \\ &= \sum_{i=1}^{n} (g_i \circ f) d(x_i \circ f) \\ &= \sum_{i=1}^{n} (g_i \circ f) \sum_{j=1}^{n} \tfrac{\partial f^i}{\partial x_j} dx_j. \end{aligned}$$

Here, the first equality is linearity, the second equality follows from the way that f^* works on functions, the third follows from from the fact that f^* commutes with derivatives of functions and the last equality follows from the definition of d.

EXAMPLE 2.2.5. If $j : \mathrm{U} \subset \mathrm{M}$ is an open subset, then $\mathrm{T}_x\mathrm{U} = \mathrm{T}_x\mathrm{M}$ for any $x \in \mathrm{U}$ and in particular, $\bigwedge^p \mathrm{T}_x^*\mathrm{M} = \bigwedge^p \mathrm{T}_x^*\mathrm{U}$. In this case, the pullback operation is rather easy, one simply takes a form on M and regards it as a form on U. To this end we write $j^*\omega = \omega|_{\mathrm{U}}$.

EXAMPLE 2.2.6 (The fundamental form on the circle revisited). Let $\iota : \mathrm{S}^1 \hookrightarrow \mathbb{R}^2$ be the usual embedding. We have implicitly seen a pullbacks: on $\mathbb{R}^2$ we can consider the form $\omega = -ydx + xdy$. Then from Example 1.0.8, we see that $\iota^*(\omega)$ defines a nowhere vanishing section on S^1. In fact, not all forms on $\mathbb{R}^2$ pulls back to something which is nowhere zero on S^1. For example, consider $\omega' = xdx + ydy$. Then it is easy to calculate that $\omega' = dg$ where

$$g : \mathbb{R}^2 \to \mathbb{R} \qquad (x, y) \mapsto \tfrac{1}{2}x^2 + y^2.$$

But $g|_{\mathrm{S}^1}$ is literally the constant function 1. Hence $\omega'|_{\mathrm{S}^1} = dg|_{\mathrm{S}^1} = d(\iota^* g) = 0$.

The following is the key lemma in extending the exterior derivative operation to the entire manifold.

LEMMA 2.2.7. *If $f : \mathrm{U} \to \mathrm{V}$ be a smooth map between opens of Euclidean space. Then $g^*d = dg^*$.*

PROOF. Via general properties of g^* and d, it suffices that they coincide on the form

$$\sum f_{i_1 \cdots i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

But, we can again check that agree on individual terms of the sum and eventually on each $f_{i_1 \cdots i_p}$ and dx_i. On functions this is the last part of Proposition 2.2.3. Now, $g^* ddx_i = 0$ on the one hand. On the other, $dg^* dx_i = dd(g^* x_i) = 0$ since $d^2 = 0$. $\square$

In this manner, if $(\mathrm{U}, \mathrm{V}, \varphi)$ is a chart around $x \in \mathrm{M}$ and ω is a form. Then $d\omega$ around x is defined to be $(\varphi^{-1})^* d\varphi^*\omega$. To prove that this is well-defined we need to check on overlaps. Let

$(\mathrm{U}', \mathrm{V}', \psi)$ be another chart around x. Then

$$\begin{aligned}(d\omega|_{\mathrm{U}\cap\mathrm{V}}) &= d(\omega|_{\mathrm{U}\cap\mathrm{V}}) \\ &= (\varphi^{-1})^* d\varphi^* \omega_{\mathrm{U}\cap\mathrm{V}} \\ &= (\varphi\psi^{-1})^*(\varphi^{-1})^* d\varphi^*(\varphi^{-1}\psi)^*\omega|_{\mathrm{U}\cap\mathrm{V}} \\ &= (\psi^{-1})^*\varphi^*(\varphi^{-1})^* d\varphi^*(\varphi^{-1})^*\psi^*\omega_{\mathrm{U}\cap\mathrm{V}} \\ &= (\psi^{-1})^* d\psi^*\omega_{\mathrm{U}\cap\mathrm{V}}.\end{aligned}$$

Having the exterior derivative, in the next example, we recover certain operations from multivariable calculus.

Example 2.2.8 (Grad, curl and div). We will recover the operations of usual multivariate calculus using the exterior derivative. In all cases we work with a 3-manifold, so that after choosing an approriate chart, we work with open subsets of $\mathbb{R}^3$, which we call U.

(Grad) Let $f : \mathrm{U} \to \mathbb{R}$ be a smooth function then

$$df = \tfrac{\partial f}{\partial x_1} dx_1 + \tfrac{\partial f}{\partial x_2} dx_2 + \tfrac{\partial f}{\partial x_3} dx_3$$

which is commonly known as the gradient.

(Curl) Let $\omega = f_1 dx_1 + f_2 dx_2 + f_3 dx_3$ be a 1-form, then $d\omega$ is equal to

$$\begin{aligned}&\left(\frac{\partial f_1}{\partial x_1}dx_1 + \frac{\partial f_1}{\partial x_2}dx_2 + \frac{\partial f_1}{\partial x_3}dx_3\right) \wedge dx_1 \\ +&\left(\frac{\partial f_2}{\partial x_1}dx_1 + \frac{\partial f_2}{\partial x_2}dx_2 + \frac{\partial f_2}{\partial x_3}dx_3\right) \wedge dx_2 \\ +&\left(\frac{\partial f_3}{\partial x_1}dx_1 + \frac{\partial f_3}{\partial x_2}dx_2 + \frac{\partial f_3}{\partial x_3}dx_3\right) \wedge dx_3\end{aligned}$$

This simplifies to:

$$\left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) dx_1 \wedge dx_2 + \left(\frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3}\right) dx_1 \wedge dx_3 + \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right) dx_2 \wedge dx_3.$$

(Div) In the homework you will be asked to obtain an expression for d of a 2-form.

In fact, the exterior derivative has a unique characterization.

Lemma 2.2.9 (Characterization of exterior derivative). *For any manifold* M*, there is a linear map*

$$d : \Omega^p(\mathrm{M}) \to \Omega^{p+1}(\mathrm{M}) \qquad p \geqslant 0$$

characterized uniquely by:

(1) when $p = 0$*, it is given by the ordinary derivative in the sense that it gives rise to the form* df *which is locally given by Example 1.0.5;*
(2) it is a linear map
(3) it satisfies

$$d(\omega \wedge \eta) = d(\omega) \wedge \eta + (-1)^{|\omega|}\omega \wedge d(\eta).$$

(4) $d^2 = 0$.

Proof. Using charts, we reduce to the case when M is an open subset of $\mathbb{R}^m$. You will be asked to verify the properties in the homework. To prove uniqueness, say D is another operation with the same property. By using charts again, we may let $\omega = \sum f_{i_1 \cdots i_p} dx_{i_1} \wedge \cdots d_{x_{i_p}}$ and show that $\mathrm{D}(\omega) = d(\omega)$. Indeed, using linearity, we can reduce to checking on $f_{i_1 \cdots i_p} dx_{i_1}$. Using the way d interacts with $\wedge$, we can check on functions and on each dx_i. Using the $p = 0$ case we see that they are the same on functions. Now, noting that $dx_i = d(x_i)$ and so $dx_i = \mathrm{D}x_i$ and so we need only compare the values of ddx_i and $\mathrm{DD}x_i$; this is both zero.

∎

Now, recall that at the very beginning of this section, we discussed the definition of $\mathbb{R}$-cdga. We have the following theorem:

THEOREM 2.2.10. *Let* M *be a manifold. Consider the pair*

$$\mathrm{dR_M} := (\Omega^*(\mathrm{M}) = \bigoplus_{p\geqslant 0} \Omega^p(\mathrm{M}), d),$$

called the ***de Rham complex*** *of* M. *Then:*

(1) $\mathrm{dR_M}$ *is an example of* $\mathbb{R}$*-cdga.*

(2) it is the unique $\mathbb{R}$*-cdga which satisfies the following universal property: if* $(\mathrm{R}^\star, d)$ *is any other* $\mathbb{R}$*-cdga, then for any commutative* $\mathbb{R}$*-algebra homomorphism* $f : \mathcal{O}(\mathrm{M}) \to \mathrm{R}^0$, *there exists a unique* $\mathbb{R}$*-cdga map*

$$f^\star : \mathrm{dR_M} \to (\mathrm{R}^\star, d)$$

such that $f^0 = f$. *This means that we have:*

(A) $\mathbb{R}$*-linear maps* $f^p : \Omega^p(\mathrm{M}) \to \mathrm{A}^p$ *for* $p \geqslant 0$*;*

(B) $f^{|\omega|+|\eta|}(\omega \wedge \eta) = f^{|\omega|}(\omega) \cdot f^{|\eta|}(\eta)$

(C) $df^n = f^{n+1}d$.

PROOF SKETCH. Given $f : \mathcal{O}(\mathrm{M}) \to \mathrm{R}^0$, then we can just set (on pure antisymmetric tensor of the form dg's and then extend linearly):

$$f^p(dg_1 \wedge \cdots \wedge dg_p) := df(g_1) \cdots df(g_p).$$

Noting, locally on M, $\Omega^n(\mathrm{M})$ is isomorphic to $\Omega^n(\mathbb{R}^m)$ and thus spanned by $dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ we can make sense of the above formula locally and then glue together locally to get the above map. Properties (A)-(C) then hold by design. □

The construction of $\mathrm{dR_M}$ is one of the major achievements in mathematics and is an object that has not only been influential in differential topology/geometry, but also in algebraic geometry and number theory. It is in this context that we will prove Stokes' theorem.

3. Integration of forms

Alright, so we ask ourselves what does it mean to take integration. We briefly recall the construction of the Riemann integral.

CONSTRUCTION 3.0.1 (The Riemann integral, sketched). . Given a rectangle

$$\mathrm{R} := [a_1, b_1] \times \cdots \times [a_m, b_m] \subset \mathrm{U},$$

we can associate to it, its volume

$$\mathrm{vol}(\mathrm{R}) := \prod_{i=1}^{m} (b_i - a_i).$$

We first define the integral of a bounded function $f : \mathrm{R} \to \mathbb{R}$.

A partition P (of length n) of the interval $[a, b]$ are numbers

$$a_0 = p_0 < p_1 \cdots < p_n = b.$$

A partition of the rectangle R consists of a partition P_i of $[a_i, b_i]$, $i = 1, \cdots, m$. This divides R into subrectangles R_j in the evident way so that $\mathrm{R} = \sqcup \mathrm{R}_j$.

Then the lower and upper sums of f are defined respectively as

$$\mathrm{L}(f, \mathrm{P}) := \sum (\inf_{\mathrm{R}_j} f) \mathrm{vol}(\mathrm{R}_j) \qquad \mathrm{U}(f, \mathrm{P}) := \sum (\sup_{\mathrm{R}_j} f) \mathrm{vol}(\mathrm{R}_j).$$

Noting that $\mathrm{L}(f, \mathrm{P}) \leqslant \mathrm{U}(f, \mathrm{P})$ we can refine the above sum by refining the partitions of rectangles as above. The lower and upper integrals of f on R is given by

$$\underline{\int}_{\mathrm{R}} f := \sup_{\mathrm{P}} \mathrm{L}(f, \mathrm{P}) \qquad \overline{\int}_{\mathrm{R}} f := \inf_{\mathrm{P}} \mathrm{U}(f, \mathrm{P})$$

We say that f is integrable on R is the two quantities agree and we denote it by

$$\int_{\mathrm{R}} f(x) dx_1 dx_2 \cdots dx_n.$$

In general if $\mathrm{A} \subset \mathrm{U}$ is any bounded set and $f : \mathrm{A} \to \mathbb{R}$, we enclose A in a rectangle R, and extend f by zero to a function $\widetilde{f} : \mathrm{R} \to \mathbb{R}$ and set

$$\int_{\mathrm{A}} f(x) dx_1 \cdots dx_n := \int_{\mathrm{R}} \widetilde{f} dx_1 \cdots dx_n.$$

CONSTRUCTION 3.0.2 (Integration of forms). Any m-form $\omega \in \Omega^m(\mathbb{R}^m)$ can be written as $f(x) dx_1 \wedge \cdots \wedge dx_m$, being the top degree differential form. For any set A we set

$$\int_{\mathrm{A}} \omega := \int_{\mathrm{A}} f(x) dx^1 \cdots dx_n.$$

REMARK 3.0.3 (Order of integration).

APPENDIX A

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