LECTURES 1-3

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"Instructions for living a life: Pay attention. Be astonished. Tell about it."
Sometimes, Mary Oliver.

1. From vector bundles to zeta functions

The Grothendieck group of a scheme is, by now, a rather ancient object. For simplicity, we assume that X is a quasi-projective scheme over a commutative ring R. For example X itself could be $\mathrm{Spec}\,\mathrm{R}$. Then $\mathrm{K}_0(\mathrm{X})$ defined by

$$\mathrm{K}_0(\mathrm{X}) := \frac{\mathbb{Z}\langle[\mathcal{E}]:\mathcal{E} \text{ is an isomorphism class of vector bundles on X}\rangle}{([\mathcal{E}]=[\mathcal{E}']+[\mathcal{E}''],0\to\mathcal{E}'\to\mathcal{E}\to\mathcal{E}''\to 0)}.$$

If $\mathrm{X} = \mathrm{Spec}\,\mathrm{R}$ is affine, then a finitely generated projective module is the same thing as a vector bundle on X and $\mathrm{K}_0(\mathrm{X})$ can be described as the group completion of the monoid of isomorphism classes of such objects: it is the initial abelian group equipped with a commutative monoid map $\mathrm{Proj}_{\mathrm{R}}^{\simeq} \to \mathrm{K}_0(\mathrm{X})$ where $\mathrm{Proj}_{\mathrm{R}}^{\simeq}$ is the monoid of isomorphism classes of finitely generated projective R-modules.

The idea of group-completion and K_0 has now pervaded mathematics so much that it is hard to think that it was once a very innovative move by Grothendieck. In essence K_0 captures the global structure of "linear algebra over X."

Example 1.0.1. Assigning a vector bundle its rank descends to a map of abelian groups

$$\mathrm{K}_0(\mathrm{X}) \xrightarrow{\mathrm{rank}} \mathrm{H}^0_{\mathrm{Zar}}(\mathrm{X};\mathbb{Z}).$$ [1]

Let $\mathrm{X} = \mathrm{Spec}\,\mathrm{F}$ a field. Then, up to isomorphism, every finite dimensional vector space over F is completely classified by its rank. This descends an isomorphism

$$\mathrm{rank} : \mathrm{K}_0(\mathrm{X}) \xrightarrow{\cong} \mathbb{Z} \qquad [\mathrm{V}] - [\mathrm{W}] \mapsto \mathrm{rank}(\mathrm{V}) - \mathrm{rank}(\mathrm{W}),$$

giving us the first calculation of a K-group.

Linear algebra is only slightly more difficult over dedekind rings.

Example 1.0.2. Given any ring R, we have a group homomorphism

$$\det : \mathrm{K}_0(\mathrm{R}) \to \mathrm{Pic}(\mathrm{R}).$$

This is the usual determinant construction: if P is a finitely generated projective module with constant rank n, then $\det(\mathrm{P}) \cong \bigwedge^n_{\mathrm{R}}(\mathrm{P})$; otherwise one has to perform this construction component-wise. In any case, the fact that the determinant descends to a map of groups boils down to the following fact: if P and Q are finitely generated projective modules of constant ranks n and m respectively then

$$\bigwedge_{\mathrm{R}}^{m+n}(\mathrm{P}\oplus\mathrm{Q}) = \bigwedge_{\mathrm{R}}^{n}(\mathrm{P}) \otimes \bigwedge_{\mathrm{R}}^{m}(\mathrm{Q}).$$

The combination of the the determinant and the previous construction defines a map

$$\mathrm{rank}\oplus\det : \mathrm{K}_0(\mathrm{Spec}\,\mathrm{R}) \to \mathrm{H}^0_{\mathrm{Zar}}(\mathrm{Spec}\,\mathrm{R};\mathbb{Z}) \oplus \mathrm{Pic}(\mathrm{R}).$$

[1] Note that this latter object is simply the ring of continuous functions from X (as its Zariski topological space) to $\mathbb{Z}$.

This map is surjective: if $\operatorname{Spec} R$ is connected there is a set-theoretic splitting given by $(m, \mathcal{L}) \mapsto [\mathcal{L}] - m[R]$. This already suggests the complexity of K_0: that it knows at least as much as the Picard group of $\operatorname{Spec} R$. Now, if R is (commutative), noetherian ring of Krull dimension one then it is a standard algebra fact that any finitely generated projective R-module of rank n is isomorphic to $P = R^{\oplus n-1} \oplus \det(P)$; for example see [Wei13, Proposition 3.4]. Therefore, the above map is an isomorphism in this particular instance.

The construction of det also extends to the world of schemes; in particular [Wei13, Proposition II.8.2.1] we see that if X is a 1-dimensional, separated, regular noetherian scheme then

$$\operatorname{rank} \oplus \det : K_0(X) \xrightarrow{\cong} H^0(X; \mathbb{Z}) \oplus \operatorname{Pic}(X).$$

The higher K-groups are more mysterious, but no less important than K_0. One standard motivation for them is actually calculational: let X be a smooth over k. It turns out that if if $X = U \cup V$ is an open cover, then we get an exact sequence

$$K_0(X) \to K_0(U) \oplus K_0(V) \to K_0(U \cap V) \to 0,$$

but the last map fails to be injective:

Remark 1.0.3. Let X be a scheme. Then we have the following exact sequence, owing to the formalism of the Picard stack

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}(X)^\times & \longrightarrow & \mathcal{O}(U)^\times \oplus \mathcal{O}(V)^\times & \longrightarrow & \mathcal{O}(U \cap V)^\times \\ \operatorname{Pic}(X) & \longrightarrow & \operatorname{Pic}(U) \oplus \operatorname{Pic}(V) & \longrightarrow & \operatorname{Pic}(U \cap V) & \longrightarrow & 0. \end{array}$$

We can use the calculation of K_0 of 1-dimensional noetherian schemes rings of Example 1.0.2 to give examples of the failure of the map $K_0(X) \to K_0(U) \oplus K_0(V)$ to be injective.

We begin with an easy non-affine example; assume that k is a field (or a regular local ring) and consider the open cover of $\mathbb{P}^1_k$ by two copies of $\mathbb{A}^1_k$. We know that $\operatorname{Pic}(\mathbb{P}^1_k) \cong \mathbb{Z}\langle \mathcal{O}(1) \rangle$. But then $\operatorname{Pic}(\mathbb{A}^1_k) \cong \operatorname{Pic}(k) \cong 0$. Hence, there is no way that the map on Picard groups is injective[2].

Let us come up with an affine example: we want to find a dedekind domain R, nontrivial invertible sheaf $\mathcal{L}$ on $\operatorname{Spec} R$ and elements $f, g \in R$ with $(f, g) = R$ and $\mathcal{L}[\frac{1}{f}], \mathcal{L}[\frac{1}{g}]$ trivial. Let k be an algebraically closed field of characteristic zero. We will consider the following affine scheme: let R be

$$R = k[x, y]/(y^2 - x^3 - 1).$$

In other words, it is the closed subscheme of $\mathbf{A}^2$ cut out by the equation $y^2 = x^3 + 1$. This is an example of a **punctured elliptic curve**; it is easy to see that $\operatorname{Pic}(R) \cong \overline{X}(k)$, the abelian group of k-points of the elliptic curve itself. Hence any non-identity point is a nontrivial element of $\operatorname{Pic}(R)$. Consider the following elements of R: $f = y - 1, g = y + 1$. In this case, we have that

$$f - g = y - 1 - (y + 1) = -2$$

so that indeed $(f, g) = R$. Consider the ideal (which corresponds to an element of the Picard group under the divisor-line bundle correspondence)

$$\mathcal{L} = (x, y + 1).$$

Geometrically, this coincides with the point $(0, -1) \in \overline{X}(k)$ which is nonzero. Now, I claim that $\mathcal{L}$ is free on the charts $\operatorname{Spec} R_{g_1}, \operatorname{Spec} R_{g_2}$. Indeed

$$\mathcal{L}|_{R_g} = (x, y + 1)[\tfrac{1}{y+1}] = (y + 1),$$

while

$$\mathcal{L}|_{R_f} = (x, y + 1)[\tfrac{1}{y-1}] = (x)$$

[2]One can also calculate that $\mathbb{Z} \oplus \mathbb{Z} \cong K_0(\mathbb{P}^1_k) \to K_0(\mathbb{A}^1_k) \oplus K_0(\mathbb{A}^1_k) \cong \mathbb{Z} \oplus \mathbb{Z}$ is a rank 1-matrix.

since

$$x^3 = 1 - y^2 = (1-y)(1+y) \Rightarrow \tfrac{x^3}{1-y} = 1 + y.$$

Therefore we see that $\mathcal{L}$ is a locally free sheaf which is not globally trivial.

Inspired by generalized cohomology theories from topology, one can imagine a sequence of groups $\{\mathrm{K}_i(\mathrm{X})\}_{i \geqslant 0}$ which rectifies this failure in the sense we get a long exact sequence

$$\begin{array}{l}
\cdots \longrightarrow \mathrm{K}_i(\mathrm{U}\cap\mathrm{V}) \longrightarrow \mathrm{K}_i(\mathrm{U}) \oplus \mathrm{K}_i(\mathrm{V}) \longrightarrow \mathrm{K}_i(\mathrm{U}\cap\mathrm{V}) \\
\cdots \longrightarrow \mathrm{K}_1(\mathrm{U}\cap\mathrm{V}) \longrightarrow \mathrm{K}_1(\mathrm{U}) \oplus \mathrm{K}_1(\mathrm{V}) \longrightarrow \mathrm{K}_1(\mathrm{U}\cap\mathrm{V}) \\
\mathrm{K}_0(\mathrm{X}) \longrightarrow \mathrm{K}_0(\mathrm{U}) \oplus \mathrm{K}_0(\mathrm{V}) \longrightarrow \mathrm{K}_0(\mathrm{U}\cap\mathrm{V}) \longrightarrow 0.
\end{array}$$

In particular, the discussion of Remark 1.0.3 suggests that K_1 should have something to do with units. Indeed, *a posteriori*, there is a determinant map

$$\det : \mathrm{K}_1(\mathrm{X}) \to \mathrm{H}^0_{\mathrm{Zar}}(\mathrm{X}; \mathbb{G}_m) \cong \mathcal{O}(\mathrm{X})^\times$$

which turns out to be an isomorphism whenever X is a semilocal ring [Wei13, Lemma III.1.4]. Eventually, the higher K-groups were first defined by Quillen in his remarkable paper [Qui10] but it was not until Thomason-Trobaugh [TT90] that we can associate to a general quasi-compact, quasi-separated scheme X, its K-groups (including negative ones!) $\{\mathrm{K}_i(\mathrm{X})\}_{i\in\mathbb{Z}}$ satisfying the Mayer-Vietoris long exact sequence as above. It is then natural to ask what information the higher K-groups contain, keeping in mind that K_0 has something to do with linear algebra.

1.1. Zeta functions and sizes of K-groups.

One of the most compelling answers to the above questions has to do with zeta functions. Computation of K-groups are few and far between, the first complete one was of finite fields and is due to Quillen himself [Qui72]:

Theorem 1.1.1. *Let $q = p^n$, then*

$$\mathrm{K}_j(\mathbb{F}_q) = \begin{cases} \mathbb{Z} & j = 0 \\ 0 & j = 2i, j < 0 \\ \mathbb{Z}/(q^i - 1) & j = 2i-1, i > 0 \end{cases}.$$

Lichtenbaum [Lic73] observed that Quillen's calculation, coupled with some other contemporaneous development due to Tate, is actually very suggestive. The following complex-valued function is a classical object.

Definition 1.1.2 (Hasse-Weil)**.** Let X be a finite type $\mathbb{Z}$-scheme of which is relatively equidimensional over $\mathbb{Z}$ of relative dimension d. We define the **Hasse–Weil zeta function** attached to X to be

$$\zeta(\mathrm{X}, s) = \prod_{x\in|\mathrm{X}|} \frac{1}{1 - \#\kappa(x)^{-s}};$$

where $|\mathrm{X}|$ is the topological space (set) of closed points on the scheme X and $\#k$ denotes the cardinality of a finite field k. Notice that the residue field $\kappa(x)$ of a closed point $x \in |\mathrm{X}|$ is also finite, since X is finite type over $\mathbb{Z}$.

In particular, we can write:

$$\zeta(\operatorname{Spec}\mathbb{F}_q, -s) = \frac{1}{1-q^s} = \tfrac{1}{|\mathrm{K}_{2s-1}(\mathbb{F}_q)|}.$$

Around the same time, Tate [Tat71] was studying K_2 in the context of reciprocity laws. Supported by some computer-assisted data by Birch [Bir71], we have the following conjecture.

Conjecture 1.1.3 (Birch-Tate conjecture)**.** *Let* F *be a totally real number field, then*

$$\zeta(\operatorname{Spec}\mathrm{F}, -1) = \pm\tfrac{|\mathrm{K}_2(\mathcal{O}_\mathrm{F})|}{|\mathrm{K}_3(\mathcal{O}_\mathrm{F})_{\mathrm{tors}}|}.$$

Tate affirmed the analog of this conjecture for F the function field of a curve over a finite field [Tat71]; see also Bass' Bourbaki talks on these matters [Bas71].

Remark 1.1.4 (The class number formula)**.** Anachronistically, we can also view Dirichlet's class number formula in the same vein as Conjecture 1.1.3. Let F be a number field and $\mathcal{O}_\mathrm{F}$ its ring of integers. We set $\zeta(\operatorname{Spec}\mathcal{O}_\mathrm{F}, 0)^*$ be the coefficients of the first non-vanishing term of a Taylor expansion of $\zeta(\operatorname{Spec}\mathcal{O}_\mathrm{F}, s)$ around zero; this helps make sense of special values of the zeta function even if ζ does vanish at zero, as is sometimes the case. The Dirichlet class number formula then states:

$$\zeta(\operatorname{Spec}\mathcal{O}_\mathrm{F}, 0)^* = -\frac{|\mathrm{K}_0(\mathcal{O}_\mathrm{F})_{\mathrm{tors}}|}{|\mathrm{K}_1(\mathcal{O}_\mathrm{F})_{\mathrm{torsion}}|}\cdot \mathrm{R}_\mathrm{F},$$

where R_F is Borel's regulator. For example, if $\mathrm{F} = \mathbb{Q}$ then

$$\mathrm{K}_0(\mathbb{Z}) \simeq \mathbb{Z} \qquad \mathrm{K}_1(\mathbb{Z}) = \{\pm 1\} \qquad \zeta(\operatorname{Spec}\mathbb{Z}; 0) = -\tfrac{1}{2}.$$

To relate this to what Dirichlet would have proved we need to know that:

(1) the torsion part of K_0 of a number ring is exactly $\operatorname{Pic}(\mathcal{O}_\mathrm{F})$ as explained in Example 1.0.2;
(2) the resolution of the "congruence subgroup problem" due to Bass-Milnor-Serre [BMS67] which implies that $\mathrm{K}_1(\mathcal{O}_\mathrm{F}) \cong \mathcal{O}_\mathrm{F}^\times$.

Extrapolating the Birch-Tate conjecture and the class number formula to higher K-groups and the zeta values $\zeta(\operatorname{Spec}\mathcal{O}_\mathrm{F}, s)$ for $s \leqslant 0$, Lichtenbaum conjectured in [Lic73, Conjecture 2.4] that there should be a relationship between these zeta values and the ratio of even and odd K-groups; we will not state the precise conjectures here but the reader is referred to [Kol04] or [EZ24, Section 3] for more recent surveys. Towards accessing these conjectures and supported by Tate's calculation of K_2 of number rings in terms of étale cohomology [Tat76], Quillen conjectured a cohomological formula for K-theory, first stated as [Lic73, Conjecture 2.5]. Later, in his 1974 ICM address in Vancouver, Quillen sharpened this to the following conjecture:

> "The work of Tate on K_2 of global fields suggests that $\mathrm{K}_n(\mathrm{A})$ might be related to the étale cohomology of Spec(A). To be more precise, one might hope to have a spectral sequence, analogous to the Atiyah-Hirzebruch spectral sequence of topological K-theory, starting with the étale cohomology groups
>
> $$\mathrm{E}_2^{p,q} = \begin{cases} 0 & q \text{ odd} \\ \mathrm{H}^p_{\text{ét}}(\mathrm{A}[\tfrac{1}{\ell}]; \mathbb{Z}_\ell(-q/2) & q \text{ even.} \end{cases}$$
>
> whose abutment would coincide with $\mathrm{K}_{-p-q}\mathrm{A}\otimes\mathbb{Z}_\ell$ at least in degrees $-p-q > 1+d$, where d is the Krull dimension of A. If A is the ring of integers in a number field, and either ℓ is odd or A is totally imaginary, this spectral sequence would degenerate, yielding cohomological formulas for the K-groups conjectured by Lichtenbaum."

We will soon make this prediction precise, but let us explain another perspective on what K-theory is.

1.2. K**-theory and algebraic cycles.** There is yet another answer to what K-theory is from a geometric viewpoint. To do so, we define a "homological" or, more precisely, a "Borel-Moore" counterpart to K-theory. If X is a noetherian scheme then set:

$$\mathrm{G}_0(\mathrm{X}) := \tfrac{\mathbb{Z}\langle [\mathcal{M}] : \mathcal{M} \text{ is an isomorphism class of a coherent } \mathcal{O}_\mathrm{X}\text{-module}\rangle}{([\mathcal{M}] = [\mathcal{M}'] + [\mathcal{M}''], 0\to\mathcal{M}'\to\mathcal{M}\to\mathcal{M}''\to 0)}.$$

Let me warn the reader that, typically, G-theory behaves quite differently from K-theory; in any case there is a natural map (whenever everything is defined) $\mathrm{K}_0(\mathrm{X}) \to \mathrm{G}_0(\mathrm{X})$ and it is an isomorphism whenever X is a separated, regular noetherian scheme [Wei13, Theorem II.8.2]. The point of working with coherent sheaves is that, we may speak of the (co-)dimension of supports of coherent sheaves[3]. We work with X a quasiprojective, smooth scheme over a field for simplicity although this is not necessary as one gleans from the hypotheses in [Stacks, Tag 0EFV].

Now, define:

$$\mathrm{F}^j\mathrm{K}_0(\mathrm{X}) = \mathrm{F}^j\mathrm{G}_0(\mathrm{X})$$

as the subgroup generated by the subset $\{[\mathcal{F}] : \mathrm{codim}_{\mathrm{X}}\mathrm{supp}\,(\mathcal{F}) \geqslant j\}$. These subgroups assemble into a decreasing filtration

$$\cdots \subset \mathrm{F}^{j+1}\mathrm{K}_0(\mathrm{X}) \subset \mathrm{F}^j\mathrm{K}_0(\mathrm{X}) \subset \cdots \mathrm{F}^0\mathrm{K}_0(\mathrm{X}) = \mathrm{K}_0(\mathrm{X}).$$

The graded ring of this filtration admits a map

$$\bigoplus_{j=0}^{\dim(\mathrm{X})} \mathrm{CH}^j(\mathrm{X}) \to \bigoplus_{j=0}^{\dim(\mathrm{X})} \mathrm{F}^j\mathrm{K}_0(\mathrm{X})/\mathrm{F}^{j+1}\mathrm{K}_0(\mathrm{X}) \tag{1.2.1}$$

given by associating to the class of $\mathrm{Z} \hookrightarrow \mathrm{X}$, the class of the structure sheaf[4] $[\mathcal{O}_{\mathrm{Z}}]$ which is a coherent $\mathcal{O}_{\mathrm{X}}$-module supported in Z. This map is surjective.

Lemma 1.2.2. *The degree i kernel of the above map is killed by $(i-1)!$. In particular the above map is an isogeny of graded groups and is isomorphism after tensoring with $\mathbb{Q}$.*

Proof. For a proof, see [Stacks, Tag 0FEV] or [Ful98, Chapters 15 and 18]. □

Lemma 1.2.2 leads to an isomorphism which is often stated as part of the Grothendieck-Riemann-Roch theorem

$$\mathrm{K}_0(\mathrm{X})_{\mathbb{Q}} \cong \bigoplus_{j=0}^{\dim(\mathrm{X})} \mathrm{CH}^j(\mathrm{X})_{\mathbb{Q}}$$

From this viewpoint, K_0 contains information about algebraic cycles on X (including torsion) but they are somehow "mixed together." This suggests that higher K-groups are also related to algebraic cycles and some sort of higher counterpart. For concreteness, we see from the discussion in Remark 1.0.3, that the group of units could be considered as being part of a theory of "higher algebraic cycles" in weight 1.

2. Motivic cohomology and algebraic K-theory

Much of this series of lectures is dedicated towards portions of the following theorem.

Theorem 2.0.1. *Let* X *be a regular, equicharacteristic scheme. Then:*

(1) *there exists a functorial, exhaustive, complete and multiplicative filtration*

$$\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(\mathrm{X}) \to \mathrm{K}(\mathrm{X}),$$

whose graded pieces are denoted by

$$\mathrm{gr}^j_{\mathrm{mot}}\mathrm{K}(\mathrm{X})[-2j] := \mathbb{Z}(j)^{\mathrm{mot}}(\mathrm{X}) \in \mathrm{D}(\mathbb{Z}),$$

called the ***motivic complexes*** *of* X.

(2) *Define the* ***motivic cohomology*** *of* X *by:*

$$\mathrm{H}^i_{\mathrm{mot}}(\mathrm{X};\mathbb{Z}(j)) = \mathrm{H}^i(\mathbb{Z}(j)^{\mathrm{mot}}(\mathrm{X})),$$

the spectral sequence resulting from the above filtration

$$\mathrm{H}^{i-j}_{\mathrm{mot}}(\mathrm{X};\mathbb{Z}(-j)) \Rightarrow \mathrm{K}_{-i-j}(\mathrm{X})$$

is strongly convergent.

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[3]Recall that the support of a quasicoherent sheaf $\mathcal{F}$ is the set of points of X such that $\mathcal{F}_x \neq 0$. If $\mathcal{F}$ is a finite type quasicoherent sheaf (hence, coherent, in noetherian settings), then the support is closed and, locally, coincides with the vanishing of the annihilator ideal [Stacks, Tag 00L2].

[4]In $\mathrm{K}_0(\mathrm{X})$, the structure sheaf is the alternating sum given by $\sum(-1)^i[\mathrm{P}_i]$ where $\mathrm{P}_\bullet \to \mathcal{O}_{\mathrm{Z}}$ is a projective resolution of $\mathcal{O}_{\mathrm{Z}}$.

(3) *The rationalized complexes* $\mathbb{Q}(j)^{\mathrm{mot}}$ *are étale sheaves. In fact, the filtration above splits rationally.*

(4) *With mod-*p^r *coefficients for any prime* p*, it étale sheafifies to*

$$\mathrm{L}_{\text{ét}}\mathbb{Z}/p^r(j)^{\mathrm{mot}}(\mathrm{X}) \simeq \begin{cases} \mathrm{R}\Gamma_{\text{ét}}(\mathrm{X};\mu_{p^r}^{\otimes j}) & \frac{1}{p} \in \mathcal{O}_{\mathrm{X}} \\ \mathrm{R}\Gamma_{\text{ét}}(\mathrm{X};\mathrm{W}_r\Omega^j_{\log})[-j] & p = 0. \end{cases}$$

(5) *the complexes* $\mathbb{Z}(j)^{\mathrm{mot}}$ *are Zariski-locally* $j+1$*-truncated so that the étale sheafification map* $\mathbb{Z}(j)^{\mathrm{mot}} \to \mathrm{L}_{\text{ét}}\mathbb{Z}(j)^{\mathrm{mot}}$ *factors through the Zariski-local truncation where we have an equivalence:*

$$\mathbb{Z}(j)^{\mathrm{mot}} \xrightarrow{\simeq} \tau^{\leqslant j+1}_{\mathrm{Zar}}\mathrm{L}_{\text{ét}}\mathbb{Z}(j)^{\mathrm{mot}}.$$

Remark 2.0.2 (Syntomic cohomology)**.** The cohomology theories appearing on the right hand side of Theorem 2.0.1(4) are the correct values of a theory of **syntomic cohomology** in these cases. One of the goals of this class is to acquaint participants with the syntomic theory.

Theorem 2.0.1 implies the Lichtenbaum-Quillen conjectures on zeta values, at least up to some difficult results on étale cohomology of global fields. Roughly speaking, Theorem 2.0.1 implies that K-theory has "enough étale/syntomic cohomology in it" in order to be related to the zeta values of low-dimensional rings (which are, after all, the subject of these conjectures).

2.1. **Historical account.** Theorem 2.0.1, as stated, is the culmination of the work of many people spanning more than 40 years of mathematics. I will try to outline my understanding of the history of this result.

(1) The first suggestion that algebraic K-theory should have something to do with étale cohomology was made by Lichtenbaum [Lic73, Conjecture 2.5] in the context of his conjectures relating K-theory to zeta values. This was reiterated by Quillen in his ICM address [Qui75] where he fleshed out the relationship in terms of a spectral sequence.
(2) As already mentioned, Tate [Tat76] established what became the relationship between motivic and étale cohomology, in weight two and over a global field. This was then extended by the seminal work of Merkurjev-Suslin [MS82] in the early 80s. More precisely they proved that the Galois symbol

$$\mathrm{K}_2(\mathrm{F})/m \to \mathrm{H}^2_{\text{ét}}(\mathrm{F};\mu_m^{\otimes 2})$$

is an isomorphism whenver m is invertible in F. Many years later, the Rost-Voevodsky theorem embarked on the same strategy as the original proof of Merkurjev-Suslin.
(3) The first lucid account, to my knowledge, of motivic cohomology is written in [BMS87]; this is a followup to Beilinson's notes [Bei87] where he laid out some expected properties of motivic cohomology in the final section. It is a provocative paper which I highly encourage everyone to read; it starts with a thought experiment: what if we knew what topological K-theory was before singular cohomology? Before Beilinson's papers appeared, however, Lichtenbaum had made some conjectures on the étale versions of the story [Lic84] in relation to zeta values at non-negative integers.
(4) Around the same time, Milne saw the logarithmic de Rham-Witt sheaves [Mil86] as motivic objects via his investigation of the special values of zeta functions over finite fields.
(5) Bloch and Kato were investigating analogs of the de Rham comparison theorem in the p-adic context; they proposed their famous conjecture in [BK86] and proved some cases of this conjecture.
(6) Bloch later defined his cycle complexes as a candidate in [Blo86] and the construction of its relationship with algebraic K-theory was sketched in a preprint with Lichtenbaum. Later, Friedlander and Suslin [FS02] globalized the Bloch-Lichtenbaum construction to smooth schemes over a field;

(7) Levine revisited Bloch's complexes [Lev94] and gave a different method for globalizing the Bloch-Lichtenbaum spectral sequence [Lev01]. The first complete account, to the instructor's knowledge, of the motivic spectral sequence is Levine's machinery of homotopy coniveau tower [Lev06, Lev08].
(8) Around the same time a young mathematician Vladimir Voevodsky had the vision to reproduce the motivic spectral sequence using his newly-minted theory of motivic homotopy theory [Voe02]. He broke down the construction of the motivic spectral sequence into a series of conjectures internal to stable motivic homotopy theory. The required conjectures were solved by Levine in [Lev08].
(9) Geisser and Levine wrote the massively influential [GL00], describing fully p-adic motivic cohomology for smooth schemes in characteristic $p > 0$; we will discuss much of this result from a modern viewpoint. One interpretation of this result is to relate Bloch's cycle complexes with its étale counterpart, at the prime p.
(10) Away from the prime, the counterpart to the above result is Rost-Voevodsky's famous theorem which resolves the Bloch-Kato, Beilinson-Lichtenbaum conjectures [Voe11, Voe03] using the machinery of motivic homotopy theory. The influence of Suslin's lectures in Luminy regarding motivic homology cannot be underestimated in this whole program.

Theorem 2.0.1 is one of the high points of the subject of algebraic K-theory. As stated, it is difficult to attack Quillen's conjecture because it posits the existence of a spectral sequence that abuts to a particular target, but *only in a range*. In fact, we know that we cannot exceed this range because of failure of K-theory to satisfy étale (hyper)descent as explained in Remark 3.2.1.

Instead, Theorem 2.0.1 makes a connection between the abstractly-defined graded pieces of the motivic filtration with étale versions of the theory — either ℓ-adic cohomology of Illisue-Milne cohomology. The latter are defined independently of algebraic K-theory and, in various context, makes explicit connection with the ζ functions of varieties [Mil86]. The rubber then meets the road in making this connection.

2.2. **What are we covering in this class?** In this this class we will cover the p-adic part of Theorem 2.0.1. We will give a somewhat self-contained proof of the following result:

Theorem 2.2.1. *Let* X *be a regular, equicharacteristic scheme in characteristic* $p > 0$ *and let* $r \geqslant 1$*; we consider*

$$\mathbb{Z}(j)^{\mathrm{mot}}_{\mathrm{X}}, \mathbb{Z}/p^r(j)^{\mathrm{mot}}_{\mathrm{X}} \in \mathrm{D}(\mathrm{X}_{\mathrm{Zar}}) \qquad j \geqslant 0:$$

as Zariski sheaves of complexes on X*; and denote by* $\epsilon : \mathrm{X}_{\mathrm{\acute{e}t}} \to \mathrm{X}_{\mathrm{Zar}}$ *the evident morphism of sites. Then:*

(1) it étale sheafifies to

$$\epsilon^*\mathbb{Z}/p^r(j)^{\mathrm{mot}}_{\mathrm{X}} \simeq \mathrm{W}_r\Omega^j_{\mathrm{X},\log}[-j].$$

(2) We have an equivalence in $\mathrm{D}(\mathrm{X}_{\mathrm{Zar}})$*:*

$$\mathbb{Z}/p^r(j)^{\mathrm{mot}}_{\mathrm{X}} \xrightarrow{\simeq} \tau^{\leqslant j}\epsilon_*\mathrm{W}_r\Omega^j_{\mathrm{X},\log}[-j].$$

In particular, for $\mathcal{O}$ *be a regular, local* $\mathbb{F}_p$*-algebra, then for all* $r \geqslant 1$ *and* $i, j \geqslant 0$*:*

$$\mathrm{H}^i_{\mathrm{mot}}(\mathcal{O}; \mathbb{Z}/p^r(j)) = \begin{cases} 0 & i \neq j \\ \mathrm{W}_r\Omega^j_{\log,\mathcal{O}} & i = j \end{cases}. \tag{2.2.2}$$

(3) The filtration in Theorem 2.2.1 coincides with the Nisnevich Postinkov filtration after modding out by p^r*.*

In particular, the vanishing of (2.2.2) is very surprising and hard to deduce from first principles. Theorem 2.2.1 itself has numerous applications and consequences, some of which we will explore throughout this class.

3. Tactical overview

To start with, we discuss some notation that we will use:

3.1. Notation.

- If $(\mathcal{C}, \tau)$ is a site, and $\mathrm{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Spt}$ is a presheaf of spectra, then we shall write $\mathcal{F}^{\tau}_{j}$ for the homotopy sheaves associated to F; this can be calculated as the τ-sheafification of the presheaf of abelian groups
$$\mathrm{U} \in \mathcal{C} \mapsto \pi_j \mathrm{F}(\mathrm{U}).$$
- The above construction is a special of the construction of a homotopy object in a t-structure. We will write $\tau^{\leqslant j}, \tau^{\geqslant j}$ for cohomological versions of the truncation functors so that we have a cofibre sequence
$$\tau^{\leqslant j} \to \mathrm{id} \to \tau^{\geqslant j+1};$$
this is the opposite convention to [Lur17, 1.2.1].
- we will elaborate more on the formalism of $\mathbb{A}^1$-invariant motivic stable homotopy theory but to appreciate the outline of the proof we need the ∞-category $\mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_{\mathrm{X}}; \mathrm{Spt})$ of $\mathbb{A}^1$-invariant, Nisnevich sheaves of spectra on a scheme X and $\mathbf{SH}(\mathrm{X})$ the Morel-Voevodsky category of motivic spectra; these are presentably symmetric monoidal ∞-categories and come equipped with an adjunction whose left adjoint is symmetric monoidal.
$$\sigma^{\infty} : \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_{\mathrm{X}}; \mathrm{Spt}) \rightleftarrows \mathbf{SH}(\mathrm{X}) : \omega^{\infty}$$
To interact with the categories involved, note that we have the symmetric monoidal Yoneda functor $h : \mathrm{Sm}_{\mathrm{X}} \to \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_{\mathrm{X}}; \mathrm{Spt})$ and the composite is denoted by
$$\mathrm{M}_{\mathrm{X}}(\mathrm{Y}) := \sigma^{\infty} h(\mathrm{Y}).$$
- The essential image
$$\sigma^{\infty}\mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_{\mathrm{X}}; \mathrm{Spt}) =: \mathbf{SH}^{\mathrm{eff}}(\mathrm{X}) \subset \mathbf{SH}(\mathrm{X})$$
is called the subcategory of **effective motivic spectra**. We can tensor the above category j-times by the **Tate object**
$$\mathbb{T}_{\mathrm{X}} := \mathrm{cof}(\mathrm{M}_{\mathrm{X}}(\mathrm{X}) \xrightarrow{\infty} \mathrm{M}_{\mathrm{X}}(\mathbb{P}^1_{\mathrm{X}})),$$
to get
$$\mathbf{SH}^{\mathrm{eff}}(\mathrm{X})(j) := \mathbf{SH}^{\mathrm{eff}}(\mathrm{X}) \otimes \mathbb{T}_{\mathrm{X}}^{\otimes j},$$
the subcategory of **j-effective motivic spectra**.

3.2. Motivating the proof. In broad strokes Theorem 2.0.1, says that there is a filtration on K-theory of a smooth k-scheme whose graded pieces are describable in syntomic terms. This should be very surprising, primarily because K-theory does not satisfy étale descent, let alone flat descent. The latter is a property that the syntomic sheaves enjoy. We offer the following standard explanation:

Remark 3.2.1 (K-theory and the Brauer group)**.** Here is a good reason why K-theory does not satisfy étale descent and yet kind of wants to; suppose that K-theory does satisfy étale *hyperdescent*. Then a reasonable construction of the motivic filtration would be just to take the the double-speed étale-Postnikov filtration. Let ℓ be a prime invertible in a field L and assume that $\mathrm{cd}_{\ell}(\mathrm{L}) = 2$ (e.g. ℓ is odd and L is a number field). Then the spectral sequence will collapse to give an isomorphism
$$\mathrm{K}_0(\mathrm{L}; \mathbb{Z}/\ell) \cong \mathrm{H}^0_{\text{ét}}(\mathrm{L}; \mathcal{K}/\ell^{\text{ét}}_0) \oplus \mathrm{H}^2_{\text{ét}}(\mathrm{L}, \mathcal{K}/\ell^{\text{ét}}_2).$$
Now, $\mathcal{K}/\ell^{\text{ét}}_0 = \mathbb{Z}/\ell$ while $\mathcal{K}/\ell^{\text{ét}}_2 \cong \mu_p$ since it fits into an exact sequence $0 \to \mathcal{K}/\ell^{\text{ét}}_2 \to \mathcal{K}^{\text{ét}}_1 \cong \mathbb{G}_m \xrightarrow{\times \ell} \mathcal{K}^{\text{ét}}_1 \cong \mathbb{G}_m \to 0$. Therefore, we conclude that $\mathrm{H}^2_{\text{ét}}(\mathrm{L}, \mu_{\ell}) \cong \mathrm{Br}(\mathrm{L})[p]$, while one expects $\mathrm{K}_0(\mathrm{L}; \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell$ via the exact sequence $\mathrm{K}_0(\mathrm{L}) \cong \mathbb{Z} \xrightarrow{\ell} \mathrm{K}_0(\mathrm{L}) \cong \mathbb{Z} \to \mathrm{K}_0(\mathrm{L}; \mathbb{Z}/\ell) \to 0$.

Nonetheless, still sticking to the same L, we have an exact sequence

$$0 \to \mathrm{K}_2(\mathrm{L})/\ell \to \mathrm{K}_2(\mathrm{L};\mathbb{Z}/\ell) \to \mathrm{L}^{\times}[\ell] \to 0.$$

Assuming that L admits an ℓ-th of unity, then the norm residue map

$$\mathrm{K}_2(\mathrm{L})/\ell \to \mathrm{H}^2_{\text{ét}}(\mathrm{L}, \mu_\ell)$$

is an isomorphism by the Merkurjev-Suslin theorem [MS82]. Therefore, $\mathrm{K}_2(\mathrm{L};\mathbb{Z}/\ell)$ contains the ℓ-torsion of the Brauer group of L. We see that the Brauer groups is indeed in K-theory but *not* in the degree predicted by étale descent.

Nonetheless, this is not at all hopeless — Beilinson taught us that we just have to cut "below the weight" of a syntomic theory in order to get a motivic theory. This is an incredibly deep insight.

Remark 3.2.2 (K-theory and algebraic cycles)**.** One way to arrive at the above formula is to take algebraic cycles into consideration. For simplicity, we work with a smooth scheme X over a field where ℓ is invertible and $\zeta_\ell \in \mathrm{F}$. We have a change-of-site spectral sequence

$$\mathrm{H}^p_{\mathrm{Zar}}(\mathrm{X}; \mathcal{H}^q(\mathbb{Z}/\ell)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(\mathrm{X}; \mathbb{Z}/\ell).$$

The work of Bloch-Ogus on Gersten resolutions for étale cohomology [BO74] shows that

$$\mathrm{H}^p_{\mathrm{Zar}}(\mathrm{X}; \mathcal{H}^q(\mathbb{Z}/\ell)) = 0$$

for $p > q$ and, furthermore,

$$\mathrm{H}^p_{\mathrm{Zar}}(\mathrm{X}; \mathcal{H}^p(\mathbb{Z}/\ell)) \cong \mathrm{CH}^p(\mathrm{X})/\ell.$$

If we consider the Beilinson-Lichtenbaum truncation at weight j, i.e., consider the complex $\tau^{\leqslant j}\mathrm{R}\epsilon_*\mathbb{Z}/\ell \in \mathrm{D}(\mathrm{X}_{\mathrm{Zar}})$, then we effectively ensure that the term $\mathrm{H}^j_{\mathrm{Zar}}(\mathrm{X}; \mathcal{H}^j(\mathbb{Z}/\ell))$ survives the descent spectral sequence and we get

$$\mathrm{H}^{2j}(\tau^{\leqslant j}\mathrm{R}\epsilon_*\mathbb{Z}/\ell(\mathrm{X})) \cong \mathrm{CH}^j(\mathrm{X})/\ell.$$

3.3. **Step 1: constructing the filtration.** The construction of the slice filtration itself is incredibly easy and works in massive generality. Let X be a scheme. Then any motivic spectrum $\mathrm{E} \in \mathbf{SH}(\mathrm{X})$ admits an exhaustive descending filtration

$$f^{\star}_{\mathrm{slice}}\mathrm{E} \to \mathrm{E}.$$

For each $j \in \mathbb{Z}$, the spectrum $f^j_{\mathrm{slice}}\mathrm{E}$ enjoys a universal property: it is the final example of a j-effective motivic spectrum over E. This ensures that the slice filtration admits good functorial properties: for example it assembles into a functor

$$\mathbf{SH}(\mathrm{X}) \to \mathbf{SH}(\mathrm{X})^{(\mathbb{Z},\geqslant)^{\mathrm{op}}}.$$

and even interacts well with multiplicative structures. We also write

$$s^j\mathrm{E} := \mathrm{cofib}(f^{j+1}_{\mathrm{slice}}\mathrm{E} \to f^j_{\mathrm{slice}}\mathrm{E})$$

for the graded pieces, or the **slices** of E.

Applying this to E = KGL, the motivic spectrum representing (homotopy) algebraic K-theory, we obtain a filtration

$$\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{KH}(\mathrm{X}) := (\omega^\infty f^{\star}_{\mathrm{slice}}\mathrm{KGL})(\mathrm{X}).$$

This is our candidate filtration. Hence our candidate theory of $\mathbb{A}^1$-invariant motivic cohomology will be declared to the graded layers:

$$\mathbb{Z}(j)^{\mathbb{A}^1}(\mathrm{X}) := \mathrm{gr}^j_{\mathrm{mot}}\mathrm{KH}(\mathrm{X})[-2j];$$

these assemble into a presheaf of graded $\mathbb{E}_\infty$-rings:

$$\{\mathbb{Z}(\star)^{\mathbb{A}^1}\} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Spt}^{\mathbb{N}^\delta})$$

Remark 3.3.1 (Comment on notation)**.** Since K-theory is not $\mathbb{A}^1$-invariant in general, we should not expect $\mathbb{Z}(\star)^{\mathbb{A}^1}$ to be the "correct" theory of motivic cohomology. Hence we refrain from the "mot" superscript. It will turn out, as we will prove over the course of this class (at least p-adically), that this is the right object for smooth schemes over a field so that $\mathbb{Z}(\star)^{\mathbb{A}^1}(\mathrm{X}) \simeq \mathbb{Z}(\star)^{\mathrm{mot}}(\mathrm{X})$ for these X's.

3.4. **Step 2: establish connectivity bounds and structural properties.** Now, we work over a perfect field. While the filtration above exists abstractly, it is quite hard to access or even say anything concrete about it. In order to proceed further, we introduce an auxilliary filtration on the ∞-category $\mathrm{Shv}_{\mathbb{A}^1}(\mathrm{Sm}_k;\mathrm{Spt})$; given an object E we have a $\mathbb{N}$-indexed filtration

$$f^{\star}_{\mathrm{uslice}}\mathrm{E} \to \mathrm{E},$$

where "uslice" stands for the "unstable slice filtration." We write

$$s^j_u\mathrm{E} := \mathrm{cofib}(f^{j+1}_{\mathrm{uslice}}\mathrm{E} \to f^{j}_{\mathrm{uslice}}\mathrm{E})$$

for the graded pieces. There is a way to study the unstable slice filtration geometrically, mainly via the purity isomorphism in motivic homotopy theory. The latter lets us conclude that if X is a smooth, irreducible k-scheme and $\mathrm{U} \subset \mathrm{X}$ is a dense open, then the map $\mathrm{U} \to \mathrm{X}$ induces an equivalence on unstable zero-slices.

The key result that one proves is:

Theorem 3.4.1 (Levine)**.** *For a perfect field k, The following diagram commutes*

$$\begin{array}{ccc} \mathbf{SH}(k) & \xrightarrow{\omega^{\infty}} & \mathrm{Shv}_{\mathbb{A}^1}(\mathrm{Sm}_k;\mathrm{Spt}) \\ \downarrow{\scriptstyle s^0} & & \downarrow{\scriptstyle s^0_u} \\ \mathbf{SH}(k) & \xrightarrow{\omega^{\infty}} & \mathrm{Shv}_{\mathbb{A}^1}(\mathrm{Sm}_k;\mathrm{Spt}). \end{array}$$

This result was conjectured by Voevodsky in [Voe02] and proved by Levine in [Lev08] via the machinery of the homotopy coniveau tower. We instead approach it via the theory of motivic infinite loop spaces a developed by Bachmann, Hoyois, Khan, Sosnilo, Yakerson and the author. Equipped with this theorem and Morel's construction of a t-structure on $\mathbf{SH}(\mathrm{X})$ [Mor05], one can prove some relatively concrete statements about the motivic filtration:

Theorem 3.4.2. *We have for any essentially smooth k-scheme X that:*

1. *the spectra $\mathbb{Z}(j)^{\mathbb{A}^1}(\mathrm{X})$ are, in fact, $\mathrm{H}\mathbb{Z}$-modules;*
2. *if L is a finitely generated field extension of k, there is a graded multiplicative map*

$$\mathrm{K}^{\mathrm{M}}_{\star}(\mathrm{L}) \to \mathrm{H}^{\star}_{\mathbb{A}^1}(\mathrm{L};\mathbb{Z}(\star)).$$

3. *The Zariski sheaves of spectra on X*

$$\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}|_{\mathrm{X}_{\mathrm{Zar}}}$$

are j-connective for all $j \geqslant 0$.

3.5. **Step 3: bring the syntomic theory into the picture.** Next, we specialize to equicharacteristic $p > 0$. Let us write $\mathbb{Z}(j)^{\mathbb{A}^1}_{\mathrm{X}} \in \mathrm{D}(\mathrm{X}_{\mathrm{Zar}})$ if we want to think of it as a Zariski sheaf on a particular X and denote by $\epsilon : \mathrm{X}_{\acute{e}t} \to \mathrm{X}_{\mathrm{Zar}}$ the usual morphism of sites. We want to produce a map of Zariski sheaves, at least whenever X is smooth over a perfect field k:

$$\mathbb{Z}(j)^{\mathbb{A}^1}_{\mathrm{X}}/p^r \to \epsilon_*\mathrm{W}_r\Omega^j_{\mathrm{X},\log}[-j], \qquad (3.5.1)$$

which are appropriately multiplicative. This will then let us factor the differential symbol in Milnor K-theory as.

$$\mathrm{K}^{\mathrm{M}}_j(\mathrm{L})/p^r \to \mathrm{H}^j_{\mathbb{A}^1}(\mathrm{L};\mathbb{Z}/p^r(j)) \to \mathrm{W}_r\Omega^j_{\mathrm{L},\log}.$$

The construction of (3.5.1) requires us to study the trace map

$$\mathrm{K} \to \mathrm{TC},$$

where TC is **topological cyclic homology**. The goal is to upgrade the trace map to a map of filtered spectra where TC is equipped with the motivic filtration of Bhatt-Morrow-Scholze [BMS19]. This is done by knowing connectivity bounds on both the filtered pieces of K-theory and TC.

3.6. **Step 5: Reduce everything to one vanishing statement and one surjectivity statement.** We now arrive at the assembly part of the proof. We will need the following geometric objects: for a field F, we have the algebraic n-algebraic simplex $\Delta^n_{\mathrm{F}} := \operatorname{Spec} \mathrm{F}[\mathrm{T}_0, \cdots, \mathrm{T}_n]/(\mathrm{T}_0 + \cdots + \mathrm{T}_n - 1)$ and its boundary divisor

$$\partial\Delta^n_{\mathrm{F}} = \operatorname{Spec} \mathrm{F}[\mathrm{T}_0, \cdots, \mathrm{T}_n]/(\mathrm{T}_0 \cdots \mathrm{T}_n(\mathrm{T}_0 + \cdots + \mathrm{T}_n - 1)) \subset \Delta^n_{\mathrm{F}}.$$

One should think of $\partial\Delta^n_{\mathrm{F}}$ as an algebraic $n-1$-sphere. Therefore, one would like a formula that imitates the "suspension isomorphism" in topology:

$$\mathrm{H}^j_{\mathrm{mot}}(\partial\Delta^{n+1}_{\mathrm{F}}; \mathbb{Z}/p(j)) \cong \mathrm{H}^j_{\mathrm{mot}}(\mathrm{F}; \mathbb{Z}/p(j)) \oplus \mathrm{H}^{j-n}_{\mathrm{mot}}(\mathrm{F}; \mathbb{Z}/p(j)).$$

This is all well and good, but to make sense of this one has to decide what motivic cohomology on a singular scheme is. It turns out that this is possible. Hence, in order to prove the Geisser-Levine theorem, say to prove the vanishing bounds of Theorem 2.2.1, one could try to do the following. Proceed by induction on weight and assume that we had settled the result for $j = 1$. Consider the j-fold multiplication map:

$$\bigwedge^j \mathrm{H}^1(\partial\Delta^{n+1}_{\mathrm{F}}; \mathbb{Z}/p(1)) \to \mathrm{H}^j(\partial\Delta^{n+1}_{\mathrm{F}}; \mathbb{Z}/p(j)).$$

By the $j = 1$ case we have $\bigwedge^j \mathrm{H}^1(\partial\Delta^{n+1}_{\mathrm{F}}; \mathbb{Z}/p(1)) \simeq \bigwedge^j \mathrm{H}^1(\mathrm{F}; \mathbb{Z}/p(1))$. Hence if j-fold multiplication map is surjective, we see that the factor which is not in degree (j, j) must be zero. However, this is difficult to verify directly. Instead, following a strategy of Suslin in a different context [Sus03] we are led to contemplate

$$\partial\widehat{\Delta}^n_{\mathrm{F}} \subset \widehat{\Delta}^n_{\mathrm{F}}$$

where the hat indicates semilocalization at vertices of the algebraic simplex.

From here, we set:

$$\mathrm{H}^i_{\mathrm{mot}}(\mathrm{X}; \mathbb{Z}/p(j)) =: \mathrm{H}^{i,j}(\mathrm{X}) \qquad \mathrm{H}^i_{\mathrm{syn}}(\mathrm{X}; \mathbb{Z}/p(j)) =: \mathrm{H}^{i,j}_{\mathrm{syn}}(\mathrm{X}),$$

and also the **Beilinson-Lichtenbaum cohomology**:

$$\mathbb{Z}/p(j)^{\mathrm{BL}}_{\mathrm{X}} := \tau^{\leqslant j}\epsilon_*\mathbb{F}_p(j)^{\mathrm{syn}}_{\mathrm{X}} \qquad \mathrm{H}^{i,j}_{\mathrm{BL}}(\mathrm{X}) := \mathrm{H}^i(\mathbb{Z}/p(j)^{\mathrm{BL}}_{\mathrm{X}}(\mathrm{X})).$$

The proof can then be organized into the following diagram (compare to [HW19, Proof of Theorem 2.37]):

(3.6.1)

$$\begin{array}{ccccccccc}
\mathrm{H}^{n,n}_{\mathrm{Z}}(\partial\Delta^p_{\mathrm{F}}) & \longrightarrow & \mathrm{H}^{n,n}(\partial\Delta^p_{\mathrm{F}}) & \longrightarrow & \mathrm{H}^{n,n}(\partial\widehat{\Delta}^p_{\mathrm{F}}) & \longrightarrow & \mathrm{H}^{n+1,n}_{\mathrm{Z}}(\partial\Delta^p_{\mathrm{F}}) & \longrightarrow & \mathrm{H}^{n+1,n}(\partial\Delta^p_{\mathrm{F}}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{H}^{n,n}_{\mathrm{BL,Z}}(\partial\Delta^p_{\mathrm{F}}) & \longrightarrow & \mathrm{H}^{n,n}_{\mathrm{BL}}(\partial\Delta^p_{\mathrm{F}}) & \longrightarrow & \mathrm{H}^{n,n}_{\mathrm{BL}}(\partial\widehat{\Delta}^p_{\mathrm{F}}) & \longrightarrow & \mathrm{H}^{n+1,n}_{\mathrm{BL,Z}}(\partial\Delta^p_{\mathrm{F}}) & \longrightarrow & \mathrm{H}^{n+1,n}_{\mathrm{BL}}(\partial\Delta^p_{\mathrm{F}}).
\end{array}$$

Then the Geisser-Levine theorem amounts to:

Theorem 3.6.2. *For any field* F *of characteristic p, the following hold:*

(1) The map $\mathrm{H}^{n,n}(\partial\widehat{\Delta}^p_{\mathrm{F}}) \to \mathrm{H}^{n,n}_{\mathrm{syn}}(\partial\widehat{\Delta}^p_{\mathrm{F}})$ *is surjective;*

(2) $\mathrm{H}^{>n,n}(\partial\widehat{\Delta}^p_{\mathrm{F}}) = 0$.

Theorem 3.6.2 is by no means easy, but it becomes clear what we are supposed to prove. The first statement is a combination of an old result of Kato [Kat82] and a newer result about left Kan extension of syntomic cohomology [AMMN20]. The second statement follows from a general result about effective motivic spectra, proved using some parts of the coniveau tower [Lev08].

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