LECTURES 10-12

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Last time, we built a multiplicative, functorial filtration

$$\mathrm{E} \in \mathbf{SH}(\mathrm{S}) \mapsto \cdots \to f^n_{\mathrm{slice}}\mathrm{E} \to f^{n-1}_{\mathrm{slice}}\mathrm{E} \cdots \to f^0_{\mathrm{slice}}\mathrm{E} \to f^{-1}_{\mathrm{slice}}\mathrm{E} \cdots =: f^{\star}_{\mathrm{slice}}\mathrm{E};$$
$$\mathbf{SH}(\mathrm{S}) \to \mathbf{SH}(\mathrm{S})^{(\mathbb{Z},\geqslant)^{\mathrm{op}}},$$

called the **slice filtration**. Taking associated graded pieces then produces another lax symmetric monoidal functor

$$\mathrm{E} \in \mathbf{SH}(\mathrm{S}) \mapsto s^{\star}_{\mathrm{slice}}\mathrm{E}(:= \mathrm{cofib}(f^{n+1}_{\mathrm{slice}}\mathrm{E} \to f^{n}_{\mathrm{slice}}\mathrm{E}));$$
$$\mathbf{SH}(\mathrm{S}) \to \mathbf{SH}(\mathrm{S})^{\mathbb{Z}^{\delta,\mathrm{op}}}.$$

We will specialize to $\mathrm{S} = \operatorname{Spec} k$ where k is a field and plug in $\mathrm{E} = \mathrm{KGL}$. But to make full use of the properties of the slice filtration, we need to construct KGL as a $\mathbb{E}_\infty$-object in $\mathbf{SH}(k)$. To state the precise universal property of KGL, we note that have an adjunction

$$\sigma^\infty : \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_{\mathrm{S}};\mathrm{Spt}) \rightleftarrows \mathbf{SH}(\mathrm{S}) : \omega^\infty;$$

such that applying the (spectral) Yoneda image of $\mathrm{Y} \in \mathrm{Sm}_{\mathrm{S}}$ gives $\mathrm{M}_{\mathrm{S}}(\mathrm{Y}) \in \mathbf{SH}(\mathrm{S})$. One should think of $\omega^\infty\mathrm{E}$ as extracting the "0-th spectrum" of the cohomology theory of E. Under certain hypotheses on E and S, ω^∞ basically "determines" the entire structure of E.

1. K-theory revisited

Throughout, k is a field. Our first goal is to explain K-theory in motivic terms. This constitutes building the motivic spectrum KGL in a manner that makes it manifestly multiplicative. This is an important step in our program — the multiplicative structure in motivic cohomology is inherited from the multiplicative structure in KGL. This is one advantage of our treatment of motivic cohomology, that it avoids some thorny multiplicative issues that would arise if it had been build via algebraic cycles.

Proposition 1.0.1. *There exists a unique $\mathbb{E}_\infty$-algebra in motivic spectra* $\mathrm{KGL} \in \mathbf{SH}(k)$ *characterized as:*

(1) KGL *satisfies* ***Bott periodicity;***
(2) there is an $\mathbb{E}_\infty$-equivalence: $\omega^\infty\mathrm{KGL} \simeq \mathrm{K} \in \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_k;\mathrm{Spt})$.

Proposition 1.0.1 is an enhancement of the construction of KGL outlined in the previous lecture and uses, crucially, $\mathbf{SH}(k)$ as symmetric monoidal ∞-category.

1.1. The multiplicative version of KGL.

Let us first explain a coherent version of Bott periodicity in algebraic geometry. Noting that $\mathrm{K} \in \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_k)$ is an $\mathbb{E}_\infty$-algebra, we can form

$$\mathrm{Mod}_{\mathrm{K}}(k) := \mathrm{Mod}_{\mathrm{K}}\mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_k;\mathrm{Spt}),$$

the ∞-category modules over K-theory. We have the free-forgetful adjunction

$$-\otimes \mathrm{K} : \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_k;\mathrm{Spt}) \rightleftarrows \mathrm{Mod}_{\mathrm{K}}(k) : \mathrm{U}.$$

The **Bott element** is the map

$$\beta_{\mathcal{O}} : \mathbb{P}^1/\infty = \mathbb{T} \to \mathrm{K}$$

classifying the Bott class $[\mathcal{O}]-[\mathcal{O}(-1)]$, noting that $\mathcal{O}\cong\mathcal{O}(-1)$ away from the point at ∞. As explained in the previous lecture it is homotopic to the Bass map. Applying the above left adjoint, we obtain a map that we abusively denote

$$\beta_{\mathcal{O}}:\mathbb{T}\otimes\mathrm{K}\to\mathrm{K}$$

in $\mathrm{Mod}_{\mathrm{K}}(k)$.

Definition 1.1.1 (Bott periodic objects)**.** Let $\mathcal{M}$ be a $\mathrm{Mod}_{\mathrm{K}}(k)$-module in presentable ∞-categories (we will be very explicit about the only example one cares about in a minute), then the subcategory

$$\mathrm{P}_{\beta_{\mathcal{O}}}\mathcal{M}\subset\mathcal{M},$$

of those $\mathrm{E}\in\mathcal{M}$ such that $\beta_{\mathcal{O}}^{*}:\mathrm{E}\to\mathrm{map}(\mathbb{T},\mathrm{E})$ is an equivalence is the category of **Bott periodic K-modules**. Here,

$$\mathrm{map}(\mathbb{T},-):\mathcal{M}\to\mathcal{M}$$

is right adjoint to the tensoring functor

$$(\mathbb{T}\otimes\mathrm{K})\otimes-:\mathcal{M}\to\mathcal{M}.$$

We want to make precise that KGL is the universal example of a Bott periodic object whose "underlying sheaf of spectra" is K-theory. To make sense of this, consider $\mathcal{M}=\mathrm{Mod}_{\mathrm{K}}(k)[\mathbb{T}^{-1}]$, the universal $\mathrm{Mod}_{\mathrm{K}}(k)$-module where the functor $-\otimes(\mathbb{T}\otimes\mathrm{K})$ acts invertibly. We have a symmetric monoidal functor

$$\Psi_k:\mathrm{Mod}_{\mathrm{K}}(k)\to\mathrm{Mod}_{\mathrm{K}}(k)[\mathbb{T}^{-1}].$$

Note that Ψ_k preserves Bott periodic objects and thus defines a symmetric monoidal functor on subcategories:

$$\Psi_k:\mathrm{Mod}_{\mathrm{K}}(k)\to\mathrm{Mod}_{\mathrm{K}}(k)[\mathbb{T}^{-1}].$$

Lemma 1.1.2. *The functor*

$$\Psi_k:\mathrm{P}_{\beta_{\mathcal{O}}}(\mathrm{Mod}_{\mathrm{K}}(k))\to\mathrm{P}_{\beta_{\mathcal{O}}}(\mathrm{Mod}_{\mathrm{K}}(k)[\mathbb{T}^{-1}]).$$

is an equivalence of symmetric monoidal ∞-category. In particular, any $\beta_{\mathcal{O}}$-periodic $\mathbb{E}_\infty$-algebra in $\mathcal{C}_k$ defines, uniquely, a $\beta_{\mathcal{O}}$-periodic $\mathbb{E}_\infty$-algebra in $\mathrm{Mod}_{\mathrm{K}}(k)[\mathbb{T}^{-1}]$.

Indeed, both satisfy the same universal properties. To relate this construction to **SH**, denote by Φ_k the inverse to Ψ_k. We have a commutative diagram

$$\begin{array}{ccc}\mathrm{P}_{\beta_{\mathcal{O}}}(\mathrm{Mod}_{\mathrm{K}}(k)) & \xleftarrow{\Phi_k} & \mathrm{P}_{\beta_{\mathcal{O}}}(\mathrm{Mod}_{\mathrm{K}}(k)[\mathbb{T}^{-1}]) \\ \downarrow & & \downarrow \\ \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_k) & \xleftarrow[\omega^\infty]{} & \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_k)[\mathbb{T}^{-1}]=\mathbf{SH}(k),\end{array} \tag{1.1.3}$$

where the vertical arrows are "forgetful" type functors; all the functors in sight are lax symmetric monoidal functors being right adjoint of symmetric monoidal functors.

We now produce KGL as an $\mathbb{E}_\infty$-algebra in $\mathbf{SH}(k)$. The object $\mathrm{K}\in\mathrm{Mod}_{\mathrm{K}}(k)$, by virtue of being a unit object, is an $\mathbb{E}_\infty$-algebra. It is also Bott periodic by the previous lecture. Hence it defines uniquely an object

$$\mathrm{KGL}\in\mathrm{P}_{\beta_{\mathcal{O}}}(\mathrm{Mod}_{\mathrm{K}}(k)[\mathbb{T}^{-1}]).$$

Applying the right vertical arrow of (1.1.3) to this object and, abusively, denoting the image of KGL under this functor we obtain a $\mathbb{E}_\infty$-algebra in KGL whose ω^∞ is K-theory. With slightly more work, one can show that the groupoid conisisting of a pairs (E,α) where $\mathrm{E}\in\mathrm{CAlg}_{\mathrm{K}}(k)$ and $\alpha:\mathrm{E}\to\omega^\infty\mathrm{KGL}$ such that E is Bott periodic is contractible, proving uniqueness.

1.2. **Bott periodicity and the slice tower.** We note that in $\mathbf{SH}(k)$ being Bott periodic means that the Bott map $\mathrm{KGL} \to \mathbb{T}^{\otimes -1} \otimes \mathrm{KGL}$ is an equivalence in $\mathbf{SH}(k)$. In particular, we also have an equivalence

$$\mathbb{T} \otimes \mathrm{KGL} \xrightarrow{\simeq} \mathrm{KGL}.$$

Therefore its values as a cohomology theory can be easily calculated:

$$\begin{aligned}
\mathrm{KGL}^{p,q}(\mathrm{X}) &\cong [\mathrm{M}_k(\mathrm{X}), \Sigma^{p,q}\mathrm{KGL}] \\
&\cong [\mathrm{M}_k(\mathrm{X}), \Sigma^{p+2q-2q,q}\mathrm{KGL}] \\
&\cong [\mathrm{M}_k(\mathrm{X})[2q-p], \mathbb{T}^{\otimes q} \otimes \mathrm{KGL}] \\
&\cong \pi_{2q-p}\omega^{\infty}\mathrm{KGL}(\mathrm{X}) \\
&= \mathrm{K}_{2q-p}(\mathrm{X}).
\end{aligned}$$

This gives us another way to describe the slices of KGL. First, we note that, from playing with adjunctions, one can prove the following

Lemma 1.2.1. *We have functorial (in* E*) equivalences:*

$$f^n_{\mathrm{slice}}(\mathbb{T} \otimes \mathrm{E}) \simeq \mathbb{T} \otimes f^{n-1}_{\mathrm{slice}}\mathrm{E} \qquad s^n_{\mathrm{slice}}(\mathbb{T} \otimes \mathrm{E}) \simeq \mathbb{T} \otimes s^{n-1}_{\mathrm{slice}}\mathrm{E}$$

Combining this with the Bott periodicity yields an equivalence

$$\mathbb{T} \otimes f^{n-1}_{\mathrm{slice}}\mathrm{KGL} \xrightarrow{\simeq} f^n_{\mathrm{slice}}\mathrm{KGL}_{\mathrm{X}}$$

and similarly for the slices. Now, set

$$\mathrm{kgl} := f^0_{\mathrm{slice}}\mathrm{KGL};$$

by the lax symmetric monoidal properties of slices, it is a $\mathbb{E}_\infty$-algebra. We have a map $\beta : \mathbb{T} \otimes \mathrm{kgl} \to \mathrm{kgl}$ which is homotopic to $\mathbb{T} \otimes \mathrm{kgl} \xrightarrow{\beta\otimes\mathrm{id}} \mathrm{KGL} \otimes \mathrm{kgl} \xrightarrow{\mathrm{act}} \mathrm{kgl}$. By construction, there is thus a commutative diagram

$$\begin{array}{ccc}
\mathbb{T} \otimes \mathrm{kgl} & \xrightarrow{\quad\simeq\quad} & f^1_{\mathrm{slice}}\mathrm{KGL}_{\mathrm{X}} \\
& {\scriptstyle\beta}\searrow \quad \swarrow{\scriptstyle\mathrm{can}} & \\
& \mathrm{kgl}_{\mathrm{X}} &
\end{array}.$$

More generally there are, for all $n \geqslant 0$, compatible such commutative diagram in which $\mathbb{T}$ is replaced by $\mathbb{T}^{\otimes n}$ and the Bott map β is replaced by the n-fold iterate $\beta^n : \mathbb{T}^{\otimes n} \otimes \mathrm{kgl} \to \mathrm{kgl}$. We thus obtain the desired equivalence between the $\mathbb{N}$-indexed part of the slice filtration on kgl:

$$f^{\star}_{\mathrm{slice}}\mathrm{kgl} := \mathrm{kgl} \leftarrow \cdots \leftarrow f^{n-1}_{\mathrm{slice}}\mathrm{kgl} \leftarrow f^n_{\mathrm{slice}}\mathrm{kgl} \leftarrow f^{n+1}_{\mathrm{slice}}\mathrm{kgl} \leftarrow \cdots$$

and the **Bott tower/filtration**

$$\mathbb{T}^{\star} \otimes \mathrm{kgl} := \mathrm{kgl} \leftarrow \cdots \leftarrow \mathbb{T}^{\otimes n-1} \otimes \mathrm{kgl} \xleftarrow{\mathrm{id}^{\otimes n-1}\otimes\beta} \mathbb{T}^{\otimes n} \otimes \mathrm{kgl} \xleftarrow{\mathrm{id}^{\otimes n}\otimes\beta} \mathbb{T}^{\otimes n+1}\mathrm{kgl} \leftarrow \cdots.$$

In particular, taking associated gradeds, we also obtain a description of the slices of kgl as Tate twists of the zero-th slice:

$$s^j_{\mathrm{slice}}\mathrm{kgl} \simeq \mathbb{T}^{\otimes j} \otimes s^0_{\mathrm{slice}}\mathrm{kgl} \qquad j \geqslant 0.$$

Remark 1.2.2 (Multiplicative structures and the Bott tower)**.** In a sense, the Bott filtration is a much more elementary object than the slice tower: it exploits Bott periodicity for algebraic K-theory and no other structural features of $\mathbf{SH}$. However, producing a coherent multiplicative structure on the Bott tower is challenging — at least the author does not know how to do this without the slice tower.

2. Motivic cohomology

Having discussed the slice filtration and KGL as a $\mathbb{E}_\infty$-algebra we can now speak of motivic cohomology as a multiplicative, graded object. By abstract nonsense, the functor

$$\omega^\infty : \mathbf{SH}(\mathrm{X}) \to \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_\mathrm{X})$$

admits a lax monoidal structure which induces a lax monoidal structure on filtered

$$\omega^\infty : \mathbf{SH}(\mathrm{X})^{(\mathbb{Z},\geqslant)^{\mathrm{op}}} \to \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_\mathrm{X})^{(\mathbb{Z},\geqslant)^{\mathrm{op}}},$$

and graded objects

$$\omega^\infty : \mathbf{SH}(\mathrm{X})^{\mathbb{Z}^{\delta,\mathrm{op}}} \to \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_\mathrm{X})^{\mathbb{Z}^{\delta,\mathrm{op}}}.$$

Construction 2.0.1 (The motivic filtration on K-theory)**.** Let k be a field. The **motivic filtration** on K-theory is defined as the $\mathbb{E}_\infty$-algebra in filtered sheaves:

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K} := \omega^\infty f^\star_{\mathrm{slice}}\mathrm{kgl} \in \mathrm{CAlg}(\mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_\mathrm{X})^{(\mathbb{N},\geqslant)^{\mathrm{op}}}).$$

Remark 2.0.2 (kgl versus KGL)**.** For any smooth k-scheme Y, $\mathrm{M}_\mathrm{S}(\mathrm{Y})$ is 0-effective by definition. Therefore

$$\omega^\infty\mathrm{kgl}(\mathrm{Y}) \simeq \omega^\infty\mathrm{KGL}(\mathrm{Y}) \simeq \mathrm{K}(\mathrm{Y}).$$

Therefore, the effective version of KGL is enough is capture the K-theory of schemes and the filtration above is $\mathbb{N}$-indexed.

Construction 2.0.3 (Motivic cohomology)**.** Let k be a field. We define a presheaf of $\mathbb{N}$-indexed graded $\mathbb{E}_\infty$-algebras

$$\mathbb{Z}(\star) : \mathrm{Sm}_k^{\mathrm{op}} \to \mathrm{grCAlg} =: \mathrm{CAlg}^{(\mathbb{N},\geqslant)^{\mathrm{op}}}.$$

as the graded presheaves obtained by:

$$\omega^\infty\mathrm{gr}^\star_{\mathrm{slice}}\mathrm{KGL}(\simeq \omega^\infty(\mathbb{T}^{\otimes\star} \otimes s^0_{\mathrm{slice}}\mathrm{kgl})) =: \mathbb{Z}(\star)[2\star] \in \mathrm{CAlg}(\mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_\mathrm{X})^{\mathbb{Z}^{\delta,\mathrm{op}}}).$$

The definition of motivic cohomology that we offer here is very abstract and so it is hard to say that it has anything to do with algebraic geometry. For example, we saw that even though higher K-groups are quite mysterious, we know that at the very least it has something to do with vector bundles; what about for motivic cohomology? We will address these issues, but one should appreciate that this definition, by virtue of being abstract, produces a very well-behaved object, admitting a very canonical multiplicative structure and has a direct relationship with algebraic K-theory.

Our immediate goals are to relate motivic cohomology to two invariants which we explained are related to K-theory at the beginning of the story: the determinant and rank.

3. $\mathbb{Z}$-linearity of motivic cohomology

We now prove the first nontrivial property of motivic cohomology: that it is not just a presheaf of spectra, but rather a presheaf valued in $\mathrm{D}(\mathbb{Z})$, i.e., it admits a $\mathbb{Z}$-linear structure. To do this, we need to get a more concrete handle on algebraic K-theory.

3.1. Representability of K-theory.

To do so we denote by

$$\mathrm{Gr}_n := \operatorname*{colim}_m \mathrm{Gr}_n(\mathbb{A}^m) \qquad \mathrm{Gr}_\infty := \operatorname*{colim}_n \mathrm{Gr}_n.$$

Here, $\mathrm{Gr}_n(\mathbb{A}^m)$ is the Grassmanian variety of n-planes in $\mathbb{A}^m$-space. By convention, a point $\mathrm{T} \to \mathrm{Gr}_n(\mathbb{A}^m)$ classifies a map of quasicoherent sheaves over T, $\mathcal{U} \to \mathcal{O}_\mathrm{T}^{\oplus m}$ such that $\mathcal{O}_\mathrm{T}^{\oplus m}/\mathcal{U}$ is locally free of rank $m-n$. The following is one of the key first results in motivic homotopy theory.

Theorem 3.1.1 (Grassmanian models for K-theory)**.** *On any regular scheme* S*, the maps of presheaves (of pointed spaces on* Sm_S*):*

$$\mathbb{Z} \times \mathrm{Gr}_\infty \xrightarrow{i} \mathbb{Z} \times \mathrm{BGL}_\infty \xrightarrow{t} \mathrm{K}_{\geqslant 0}.$$

are $\mathrm{L}_{\mathrm{Zar},\mathbb{A}^1}$*-equivalences.*

Proof. We prove this result in steps. Without loss of generality[1], it suffices to evaluate the above map of presheaves on R a local ring of a smooth S-scheme; actually we only need to guarantee $\mathrm{K}_0(\mathrm{R}) = \mathbb{Z}$ so that $(\coprod_{n\in\mathbb{N}} \mathrm{BGL}_n)^{\mathrm{grp}} \simeq \mathrm{K}_{\geqslant 0}(\mathrm{R})$.

First, the reader should be warned that $\mathbb{Z} \times \mathrm{BGL}_\infty$ does not come equipped with an evident monoid structure; even worse: it is not even a presheaf of H-spaces when one passes to the homotopy category. Indeed, π_1 of this presheaf of spaces is GL_∞ which is evidently not an abelian group, but π_1 of any monoid must be abelian. Therefore, there is no chance that t is an equivalence without something like $\mathrm{L}_{\mathbb{A}^1}$.

We now use Theorem 3.1.3. Applying $\mathrm{L}_{\mathbb{A}^1}$ to the comparison map $\mathbb{Z}\times\mathrm{BGL}_\infty \to (\coprod_{n\in\mathbb{N}} \mathrm{BGL}_n)^{\mathrm{grp}}$ and evaluating at R is given by the colimit of the map

$$\mathbb{Z} \times \mathrm{BGL}_\infty(\mathrm{R}[\Delta^\bullet]) \xrightarrow{\alpha^\bullet} (\coprod_n \mathrm{BGL}_n(\mathrm{R}[\Delta^\bullet]))^{\mathrm{gp}}.$$

We claim that this colimit is an equivalence. Nikolaus' refinement of the group completion theorem [Nik17, Proposition 6] states that it is indeed an equivalence as soon as $\pi_1(\mathrm{colim}_{\Delta^{\mathrm{op}}} \mathbb{Z} \times \mathrm{BGL}_\infty(\mathrm{R}[\Delta^\bullet]), x)$ is abelian at any base point. Since sifted colimits commute with products, the connected components of the geometric realization is just $\mathbb{Z}$ and π_n at any connected component are all isomorphic via "translation" for any $n \geqslant 1$. Therefore, we need only prove that $\pi_1(\mathrm{colim}_{\Delta^{\mathrm{op}}} \mathrm{BGL}(\mathrm{R}[\Delta^\bullet], *)$ is abelian. By general properties of sifted colimits[2], we see that

$$\pi_1(\operatorname*{colim}_{\Delta^{\mathrm{op}}} \mathrm{BGL}(\mathrm{R}[\Delta^\bullet], *) \cong \mathrm{coeq}(\mathrm{GL}(\mathrm{R}[t]) \rightrightarrows \mathrm{GL}(\mathrm{R})),$$

where the maps are given by setting $t = 0$ and $t = 1$. Whitehead's theorem reviewed in Theorem A.0.2 then let us conclude: since the commutator of GL(R) are given by the elementary matrices, it suffices to observe that any elementary matrix can be homotoped to the identity.

Now, to prove that i is an equivalence. Just as in topology, we have the Stiefel scheme of frames, $\mathrm{St}_n(\mathbb{A}^m)$ whose points $\mathrm{T} \to \mathrm{St}_n(\mathbb{A}^m)$ consists of pairs $\mathcal{U} \to \mathcal{O}_{\mathrm{T}}^{\oplus m}$ a point of the Grassmanian and $\mathcal{O}_{\mathrm{T}}^{\oplus m-n} \to \mathcal{O}_{\mathrm{T}}^{\oplus m} \to \mathcal{O}_{\mathrm{T}}^{\oplus m}/\mathcal{U}$ an epimorphism. We have the projection map $\mathrm{St}_n(\mathbb{A}^m) \to \mathrm{Gr}_n(\mathbb{A}^m)$ which is evidently a GL_m-torsor. This map is explicitly given by the following quotients (taken in stacks, but the resulting objects are schemes because of freeness of the actions):

$$\mathrm{St}_n(\mathbb{A}^m) = [\mathrm{GL}_{n+m}/\mathrm{GL}_n] \to \mathrm{Gr}_n(\mathbb{A}^m) = [\mathrm{GL}_{n+m}/\mathrm{GL}_n \times \mathrm{GL}_m].$$

Now, the key observation is that because any GL_r-torsor which is étale locally trivial is also Zariski locally trivial (by Hilbert theorem 90), the above map identifies with the following Zariski sheaf quotients:

$$\mathrm{St}_n(\mathbb{A}^m) = \mathrm{L}_{\mathrm{Zar}}(\mathrm{GL}_{n+m}/\mathrm{GL}_n) \to \mathrm{Gr}_n(\mathbb{A}^m) = \mathrm{L}_{\mathrm{Zar}}(\mathrm{GL}_{n+m}/\mathrm{GL}_n \times \mathrm{GL}_m).$$

One then shows that $\mathrm{St}_\infty(\mathbb{A}^m) \simeq \mathrm{colim}_n \mathrm{St}_n(\mathbb{A}^m)$ is $\mathbb{A}^1$-contractible (using the same arguments as in topology) and thus it is a Zariski sheaf with free GL_m-action whose quotient is $\mathrm{Gr}_n(\mathbb{A}^m)$. This then gives us the Zariski-local identification between the infinite Grassmanian Gr_n and BGL_n.

□

We have used the following construction in the above proof.

Construction 3.1.2. Consider the following cosimplicial scheme

$$\Delta^\bullet : [n] \mapsto \Delta^n := \mathrm{Spec}\, \mathbb{Z}[\mathrm{T}_0, \mathrm{T}_1, \cdots, \mathrm{T}_n]/(\sum_{i=0}^n \mathrm{T}_i = 1)(\cong \mathbb{A}^n_{\mathbb{Z}}).$$

[1]To be precise, we need to know two things: 1) that the presheaves involved are locally finitely presented/finitary and that 2) the Zariski site of any regular scheme is hypercomplete. The latter is not quite true but it is true if such a scheme is finite dimensional but using 1) we may places ourselves in such a situation. In fact this hypercompleteness assertion is true without regularity hypotheses [CM19].

[2]On the lowest homotopy group, it is calculated as a coequalizer.

The point of writing $\mathbb{A}^n_{\mathbb{Z}}$ in this manner is that it naturally acquires faces and degeneracies in the obvious way. For example, for each fixed $n \geqslant 1$, looking at

$$\mathrm{V}(\mathrm{T}_i) \hookrightarrow \Delta^n \qquad i = 0, \cdots, n,$$

defines $q+1$ divisors, abstractly isomorphic to $\Delta^{n-1} \cong \mathbb{A}^{n-1}$. We call an *arbitrary intersections* of subschemes of this form (for any $n \geqslant 1$) the **faces**. For any scheme S, we set

$$\Delta^{\bullet}_{\mathrm{S}} := \Delta^{\bullet} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{S},$$

and we speak of the faces of $\Delta^{\bullet}_{\mathrm{S}}$ in the same manner.

The following result makes $\mathrm{L}_{\mathbb{A}^1}$ a special kind of localization.

Theorem 3.1.3. *The endofunctor* $\mathrm{L}_{\mathbb{A}^1} : \mathrm{PSh}(\mathrm{Sm}_{\mathrm{S}}) \to \mathrm{PSh}(\mathrm{Sm}_{\mathrm{S}})$ *identifies with the endofunctor*

$$\mathrm{X} \mapsto \operatorname*{colim}_{\Delta^{\mathrm{op}}} \mathrm{X}(- \times \Delta^{\bullet})$$

Proof sketch. The key point is to observe that $\mathrm{L}_{\mathbb{A}^1}\mathrm{X}$ is $\mathbb{A}^1$-invariant. To see this we can write down, for any ring R (in fact for any scheme), an explicit simplicial homotopy

$$h^{\bullet} : \mathrm{R}[t][\Delta^{\bullet}] \to \mathrm{R}[t][\Delta^{\bullet+1}]$$

between the identity map of $\mathrm{R}[t][\Delta^{\bullet}]$ and the map $\mathrm{R}[t][\Delta^{\bullet}] \xrightarrow{t=0} \mathrm{R}[\Delta^{\bullet}] \to \mathrm{R}[t][\Delta^{\bullet}]$. The maps h^i is given by

$$h^i : f \mapsto \begin{cases} \sigma^i(f) & f \in \mathrm{R}[\Delta^n],\ \sigma^i \text{ is the degeneracy} \\ t(\mathrm{T}_{i+1} + \cdots + \mathrm{T}_{n+1}) & f = t, \end{cases}$$

and extended to a map out of $\mathrm{R}[\Delta^{\bullet}][t]$ by universal properties. Taking geometric realization of $\mathrm{X}(\mathrm{R}[\Delta^{\bullet}])$ converts these simplicial homotopies to an equivalence of spaces. □

Since sifted colimits preserve products, we have the following:

Corollary 3.1.4. *The endofunctor* $\mathrm{L}_{\mathbb{A}^1}$ *preserves products. In particular, it preserves presheaves of commutative monoids.*

3.2. **Birational motives.** To give $s^{\star}\mathrm{kgl}$ a $\mathbb{Z}$-linear structure, we need only prove that $s^0\mathrm{kgl}$ admits a $\mathbb{Z}$-linear structure. To do so, we consider the following map in the world of $\mathbb{A}^1$-invariant Nisnevich sheaves on Sm_k

$$\mathrm{K} \simeq \omega^{\infty}\mathrm{kgl} \to \omega^{\infty}s^0\mathrm{kgl}; \tag{3.2.1}$$

via Theorem 3.1.1 we have the projection map $\mathrm{K} \to \mathrm{L}_{\mathbb{A}^1,\mathrm{Zar}}(\mathbb{Z}\times\mathrm{Gr}_{\infty}) \to \mathrm{L}_{\mathbb{A}^1,\mathrm{Zar}}\mathbb{Z} \simeq \mathrm{R}\Gamma_{\mathrm{Zar}}(-;\mathbb{Z})$; this latter map is evidently multiplicative: it is simply the rank map. We now ask if there is a multiplicative factorization of (3.2.1) through the latter projection. Somewhat surprisingly, it will boil down to birational geometry, in particular the rationality of the grassmanians.

Definition 3.2.2. Consider the collection

$$\mathrm{Bir}_k = \{\Sigma^{\infty}(f_+) : f : \mathrm{U} \hookrightarrow \mathrm{X} \text{ is a dense open, X is irreducible}\} \subset \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_k)^{\Delta^1}.$$

We have the localization endofunctor

$$\mathrm{L}_{\mathrm{bir}} : \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_k)^{\Delta^1} \to \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_k)^{\Delta^1},$$

which formally inverts Bir_k. We say that a map $f : \mathrm{E} \to \mathrm{F}$ in $\mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_k)$ is a **birational equivalence** if $\mathrm{L}_{\mathrm{bir}}f$ is an equivalence.

Remark 3.2.3. The localization $\mathrm{L}_{\mathrm{bir}}$ is a monoidal localization: this boils down to the fact that $f \times \mathrm{id}$ is birational whenever f is.

The following Lemma is easy given the relative purity theorem (which we will discuss next)

Lemma 3.2.4. *Let k be a perfect field. For $\mathrm{E} \in \mathbf{SH}(k)$, the spectrum $\omega^\infty s^0\mathrm{E}$ is $\mathrm{L}_{\mathrm{bir}}$-local. In particular, if $f : \mathrm{F} \to \omega^\infty s^0\mathrm{E}$ is a morphism in $\mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_k)^{\Delta^1}$ and $\mathrm{F} \to \mathrm{G}$ is a birational equivalence, then there is a unique factorization of f through G. If $\mathrm{F} \to \mathrm{G}$ is a map of $\mathbb{E}_\infty$-algebras, then so is the factorization.*

Lemma 3.2.4 is the prototype of what will happen next in this class: we will try to control the slice filtration using birational geometry, bridged by motivic infinite loop space theory. The way one should think about Lemma 3.2.4 is that $\omega^\infty s^0\mathrm{E}$ is a birational invariant; we will soon explore the invariant nature of $\omega^\infty s^j\mathrm{E}$ for all E. First, we state the relative purity theorem of Morel and Voevodsky.

Theorem 3.2.5 (Relative motivic purity)**.** *Let* S *be a base scheme and $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ is a closed immersion of smooth* X*-schemes. Then there is a canonical $\mathrm{L}_{\mathrm{Zar},\mathrm{Nis}}$-equivalence*

$$\frac{\mathbb{V}(\mathcal{N}_i)}{\mathbb{V}(\mathcal{N}_i)\smallsetminus 0_{\mathrm{Z}}} \simeq \frac{\mathrm{X}}{\mathrm{X}\smallsetminus \mathrm{Z}}.$$

We will discuss Theorem 3.2.5 in the next class; take it for granted in these set of notes.

Proof of Lemma 3.2.4. The map $f : \mathrm{F} \to \omega^\infty s^0\mathrm{E}$ is adjoint to a map $\sigma^\infty\mathrm{F} \to s^0\mathrm{E}$. The functor σ^∞ preserves colimits, therefore if the following statement proves the claim: if $\mathrm{U} \hookrightarrow \mathrm{X}$ is an open immersion where X is a smooth, connected scheme, then

$$\mathrm{cofib}(\mathrm{M}_k(\mathrm{U}) \to \mathrm{M}_k(\mathrm{X})) \in \mathbf{SH}^{\mathrm{eff}}(k)(1).$$

For then,

$$\mathrm{Maps}(\mathrm{M}_k(\mathrm{U}), s^0\mathrm{E}) \simeq \mathrm{Maps}(\mathrm{M}_k(\mathrm{X}), s^0\mathrm{E})$$

for all E.

Let $\mathrm{Z} \hookrightarrow \mathrm{X}$ be the complement of U. Since k is perfect we may stratify Z:

$$\emptyset \subset \mathrm{Z}^n \subset \cdots \subset \mathrm{Z}^1 \subset \mathrm{Z}^0 = \mathrm{Z}$$

such that $\mathrm{Z}^k \smallsetminus \mathrm{Z}^{k-1}$ is smooth. In this way, we may assume that U has a smooth complement so that we may appeal to Theorem 3.2.5 and conclude that

$$\mathrm{cofib}(\mathrm{M}_k(\mathrm{U}) \to \mathrm{M}_k(\mathrm{X})) \simeq \mathrm{cofib}(\mathrm{M}_k(\mathbb{V}(\mathcal{N}_i) \smallsetminus 0_{\mathrm{Z}}) \to \mathbb{V}(\mathcal{N}_i)).$$

If $\mathcal{N}_i$ was trivial of rank $c \geqslant 1$, then the latter is equivalent to $\mathbb{T}^c \otimes \mathrm{M}_k(\mathrm{Z})$ which is indeed in $\mathbf{SH}^{\mathrm{eff}}(k)(\geqslant 1)$. We then use Zariski descent to reduce to this case. □

We explain some easy structural properties of motivic cohomology.

Theorem 3.2.6. *Let k be a perfect field. There is a multiplicative map of presheaves on Sm_k:*

$$\mathrm{R}\Gamma_{\mathrm{Zar}}(-;\mathbb{Z}) \to \mathbb{Z}(0)^{\mathrm{mot}}.$$

In particular, motivic cohomology promotes to a functor

$$\mathbb{Z}(\star) : \mathrm{Sm}_k^{\mathrm{op}} \to \mathrm{grCAlg}(\mathrm{D}(\mathbb{Z})).$$

Proof. Theorem 3.1.1 implies that the rank map $\mathrm{K}_{\geqslant 0} \to \mathrm{R}\Gamma_{\mathrm{Zar}}(-;\mathbb{Z}) \simeq \mathrm{R}\Gamma_{\mathrm{Nis}}(-;\mathbb{Z})$ is $\mathrm{L}_{\mathrm{bir}}$-local because the infinite Grassmanian is a rational variety. Therefore, the map $\mathrm{K}_{\geqslant 0} \to \omega^\infty s^0_{\mathrm{slice}}\mathrm{KGL}$ factors through the rank map uniquely. The result then follows from Lemma 3.2.4. □

3.3. **The unstable slice filtration.** We now explain how to contextualize the theory of birational motives. The starting point is an "unstable" analog of the slice filtration:

Definition 3.3.1. Define

$$\mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_k;\mathrm{Spt})(n) := \mathbb{T}^{\otimes n} \otimes \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_k;\mathrm{Spt}) \qquad n \geqslant 0$$

so that we have subcategories

$$\cdots \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_k;\mathrm{Spt})(n) \subset \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_k;\mathrm{Spt})(n-1) \subset \cdots \subset \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_k;\mathrm{Spt})(0) = \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_k;\mathrm{Spt}).$$

By the same reasoning as in the case of $\mathbf{SH}(k)$ we have a lax symmetric monoidal functor

$$\mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_k;\mathrm{Spt}) \to \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_k;\mathrm{Spt})^{(\mathbb{Z},\geqslant)^{\mathrm{op}}},$$

$$\mathrm{E} \in \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_k;\mathrm{Spt}) \mapsto \cdots \to f^n_{\mathrm{uslice}}\mathrm{E} \to f^{n-1}_{\mathrm{uslice}}\mathrm{E} \cdots \to f^0_{\mathrm{uslice}}\mathrm{E}.$$

called the **unstable slice filtration**.

The unstable slice filtration is somewhat more concrete than its stable counterpart because it admits a nice geometric interpretation. We begin by recalling the following easy definition of codimension in algebraic geometry.

Definition 3.3.2. Let $x \in \mathrm{X}$. Then

$$\mathrm{codim}_{\mathrm{X}}(x) = \dim(\mathcal{O}_{\mathrm{X},x}).$$

We will use the following often:

Lemma 3.3.3 (Codimension formula)**.** *Let $f : \mathrm{X} \to \mathrm{Y}$ be a flat morphism of locally noetherian schemes and $x \in \mathrm{X}$ then*

$$\mathrm{codim}_{\mathrm{X}}(x) = \mathrm{codim}_{\mathrm{Y}}(f(x)) + \mathrm{codm}_{\mathrm{X}_{f(x)}}(x).$$

The next definition refines the usual notion of a birational equivalence.

Definition 3.3.4 (n-birational equivalences)**.** Let $n \geqslant 0$, then an open immersion $\mathrm{U} \hookrightarrow \mathrm{X}$ of schemes is said to be a n**-birational** if for any $x \in \mathrm{X}$ with $\mathrm{codim}_{\mathrm{X}}(x) \leqslant n$.

Via the usual formalism, we have the endofunctor of n-birational localization

$$\mathrm{L}^n_{\mathrm{bir}} : \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_k;\mathrm{Spt}) \to \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_k;\mathrm{Spt}).$$

Using the relative purity theorem one can prove:

Proposition 3.3.5. *Let k be perfect. The following subcategories of $\mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_k;\mathrm{Spt})$ are equivalent:*

(1) the subcategory of $\mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_k;\mathrm{Spt})$ generated under colimits and retracts by objects of the form $\mathrm{cofib}(f)$ where $\mathrm{L}^n_{\mathrm{bir}}(f)$ is an equivalence.
(2) $\mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_k;\mathrm{Spt})(n)$.

Furthermore, for each $\mathrm{E} \in \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_k;\mathrm{Spt})$ we have a fibre sequence

$$f^n_{\mathrm{uslice}}\mathrm{E} \to \mathrm{E} \to \mathrm{L}^{n-1}_{\mathrm{bir}}\mathrm{E} \qquad n \geqslant 0.$$

4. Connectivity bound

We will work towards the following nontrivial connectivity bound which is necessary for our approach to the Geisser-Levine theorem.

Theorem 4.0.1. *Let k be a field. Then*

$$\mathrm{Fil}^j_{\mathrm{mot}}\mathrm{K}|_{\mathrm{Sm}_k} \in \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)_{\geqslant j}.$$

Concretely: this means that for any $\mathrm{X} \in \mathrm{Sm}_k$ we have the following vanishing bound:

$$\pi_n(\mathrm{Fil}^j_{\mathrm{mot}}\mathrm{K}(\mathrm{X})) = 0 \qquad n \leqslant j - 1 - \dim(\mathrm{X}).$$

This result is not at all clear because the slice filtration is a rather abstract construction.

4.1. **Reductions.** To prove Theorem 4.0.1, we will have to prove a much more general statement connectivity statement about motivic spectra and its slices. It is given as follows:

Theorem 4.1.1. *Let k be a perfect field. Fix $\mathrm{E} \in \mathbf{SH}(k)$ and $q \geqslant 0$. Suppose that for all $p < q$ we have that*

$$\underline{\pi}_{p,q}^{\mathrm{Nis}}\mathrm{E} \simeq 0,$$

then

$$\underline{\pi}_{<0,0}^{\mathrm{Nis}} f^q(\mathrm{E}) \simeq 0.$$

The proof of Theorem 4.1.1 follows from a statement about $\mathbb{A}^1$-invariant Nisnevich sheaf whose proof relies on a result of Morel's and a conjecture about ω^∞ and f^q made by Voevodsky, and later proved by Levine. The former is the following statement.

Lemma 4.1.2 (Connectivity criterion for unstable j-effectives)**.** *Let k be a perfect field. Let $\mathrm{E} \in \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_k)(j)$. If $\underline{\pi}_{<0}^{\mathrm{Nis}}\Omega^j_{\mathbb{G}_m}\mathrm{E} \simeq 0$, then $\underline{\pi}_{<0}^{\mathrm{Nis}}\mathrm{E} \simeq 0$.*

The key result which underlies Lemma 4.1.2 is the following theorem due to Morel.

Theorem 4.1.3 (Morel's connectivity theorem)**.** *Let k be a perfect field and let E be an $\mathbb{A}^1$-invariant Nisnevich sheaf on Sm_k. Then the following are equivalent:*

(1) E is Nisnevich-locally j-connective;
(2) $\pi_n(\mathrm{E}(\mathrm{L})) = 0$ for all $n \leqslant j - 1$ and L a finitely generated field extension of k.

Theorem 4.1.3 falls within the tradition of Gersten's conjecture within algebraic K-theory. The underlying philosophy is that $\mathbb{A}^1$-invariance and Nisnevich descent is enough to guarantee that E is controlled by its values on fields. The next result sounds somewhat technical, but is rather deep conjecture of Voevodsky's.

Theorem 4.1.4 (Levine)**.** *For a perfect field k, The following diagram commutes*

$$\begin{array}{ccc} \mathbf{SH}(k) & \xrightarrow{\omega^\infty} & \mathrm{Shv}_{\mathbb{A}^1}(\mathrm{Sm}_k;\mathrm{Spt}) \\ {\scriptstyle s^0}\downarrow & & \downarrow{\scriptstyle s^0_u} \\ \mathbf{SH}(k) & \xrightarrow{\omega^\infty} & \mathrm{Shv}_{\mathbb{A}^1}(\mathrm{Sm}_k;\mathrm{Spt}). \end{array}$$

Theorem 4.1.4 asserts that taking slices commutes with taking "$\mathbb{G}_m$-infinite loop spaces." We will motivate this statement and give its proof in a later lecture. Assuming Theorem 4.1.4 and Lemma 4.1.2 we proceed as follows:

Proof of Theorem 4.1.1. Our goal is to prove that the $\mathbb{A}^1$-invariant Nisnevich sheaf $\omega^\infty f^q\mathrm{E}$ is connective in the Nisnevich t-structure. By Theorem 4.1.4 we deduce that $\omega^\infty f^q\mathrm{E} \in \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_k)(q)$. Hence, by Lemma 4.1.2, to prove that it is connective in the Nisnevich t-structure it suffices to prove this after applying $\Omega^q_{\mathbb{G}_m}$ to $\omega^\infty f^q\mathrm{E}$. Let L be a finitely generated field extension L over k. Then, by adjunction,

$$\underline{\pi}_n^{\mathrm{Nis}}(\Omega^q_{\mathbb{G}_m}\omega^\infty f^q\mathrm{E})(\mathrm{L}) \simeq \pi_n(\mathrm{Maps}_{\mathbf{SH}(k)}(\mathbb{G}_m^{\otimes q}[n], f^q(\mathrm{E})) \simeq \pi_n(\mathrm{Maps}_{\mathbf{SH}(k)}(\mathbb{G}_m^{\otimes q}[n], \mathrm{E}),$$

which is zero by the assumption on E. Via Morel's Theorem 4.1.3, we conclude that $\Omega^q_{\mathbb{G}_m}\omega^\infty f^q\mathrm{E}$ is indeed connective. □

Proof of Theorem 4.0.1. Apply Theorem 4.1.4 to $\mathrm{E} = \mathrm{KGL}[-q]$ by noting that

$$\underline{\pi}_{p,q}^{\mathrm{Nis}}\mathrm{KGL} = \underline{\pi}_{p-2q}\mathrm{K},$$

which is zero when $p - 2q < 0$ because all schemes in sight are regular. □

Appendix A. Some linear algebra

We have indicated the origins of K_2, now we explain K_1. The group $GL(R)$ is the group of matrices with 1's on the diagonal but where there are only finitely many non-zero entries. Here is the definition of K_1:

Definition A.0.1. [Bass-Schaunel] Let R be an associative, unital ring. Then $K_1(R) := GL(R)/[GL(R), GL(R)]$.

To get a better handle on the commutator subgroup, we recall that the elementary matrix $e_{ij}(x)$ where $x \in R$ is the matrix in $GL(R)$ where the only nontrivial spot away from the diagonal is ij where the entry is x. We set $E(R) \subset GL(R)$ the subgroup generated by elementary matrices. Intuitively, $E(R)$ is the subset of $GL(R)$ of those matrices which may be reduced to the identity matrix using only row operations.

Lemma A.0.2 (Whitehead's Lemma). *Let* R *be an associative, unital ring. Then*

$$E(R) = [GL(R), GL(R)]$$

Proof. For $E(R) \subset [GL(R), GL(R)]$ one can calculate

$$e_{ij}(x) = [e_{ik}(x), e_{kj}(1)].$$

Now, Whitehead observed that one can write

$$[g, h] = \begin{bmatrix} g & 0 \\ 0 & g^{-1} \end{bmatrix} \begin{bmatrix} h & 0 \\ 0 & h^{-1} \end{bmatrix} \begin{bmatrix} (hg)^{-1} & -1 \\ 0 & hg \end{bmatrix}.$$

Note that each the terms in the product are indeed in $E(R)$. □

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