LECTURES 4-6

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All schemes that appear in these lectures are quasicompact and quasiseparated.

1. Milnor K-theory

We might as well start in degree two. The purpose of what follows is not to give a full proof of quadratic reciprocity but, rather, to familiarize ourselves with some calculations in Milnor K-theory. Let $a \in \mathbb{Z}$. Suppose we are interested in the solution of the equation

$$f_a(n) = n^2 - a.$$

Number theorists and, arguably, homotopy theorists would approach this by asking for solutions modulo primes p for every prime. We say that a is a **quadratic residue modulo** p if $f_a(n)$ has a mod-p solution. The law of **quadratic reciprocity** finds a certain interesting pattern between the solutions of $f_q(n)$ modulo p and the solutions of $f_p(n)$ modulo q whenever q is a prime. Here is a sample:

Example 1.0.1. Let $q = 5$ and $p = 11$. Then

$$4^2 = 16 \equiv 5 \mod 11.$$

Hence 5 is a quadratic residue modulo 11. On the other hand,

$$1 = 1^2 \equiv 11 \mod 5,$$

so that 11 is a quadratic residue modulo 5. From this, one might guess that p is a quadratic residue modulo q if and only if q is a quadratic residue modulo p. For another example consider $q = 3$ and $p = 7$. Then, modulo 7 we have:

$$1^2 \equiv 1 \qquad 2^2 \equiv 4 \qquad 3^3 \equiv 2 \qquad 4^2 \equiv 2 \qquad 5^2 \equiv 4 \qquad 6^2 \equiv 1.$$

So, from checking all of this, we see 3 is not a quadratic residue modulo 7. On the other hand,

$$7 \equiv 1 = 1^2 \mod 3,$$

so that 7 is a quadratic residue modulo 3. Hence the naive guess is not quite right.

Checking bigger primes get more and more unwieldy. For example, we can see what happens $a = 5$ and $p = 13$. The only possible residues modulo 5 are

$$1^2 \equiv 1 \qquad 2^2 \equiv 4 \mod 5.$$

Since $13 \equiv 3 \mod 5$, it cannot be a residue modulo 5. On the other hand, one can also show that the solutions to $f_5(n) = n^2 - 5$ is not divisible by 13. We will explain a method to do this soon.

To proceed, we define the **Legendre symbol**; let $p \geqslant 3$ be an odd prime and $a \in \mathbb{Z}$:

$$(a/p)_{\mathrm{L}} = \begin{cases} 1 & p \text{ does not divided } a \text{ and } a \text{ is a quadratic residue modulo } p \\ -1 & p \text{ does not divided } a \text{ and } a \text{ is } \textit{not} \text{ a quadratic residue modulo } p \\ 0 & p \text{ divides } a. \end{cases}$$

There is actually a nicer expression for the Legendre symbol, which will feature later and is due to Euler.

Lemma 1.0.2 (Euler)**.** *Let $p \geqslant 3$ be an odd prime and $a \in \mathbb{Z}$. Then*

$$(a/p)_{\mathrm{L}} \equiv a^{\frac{p-1}{2}} \mod p.$$

The big theorem that one has in this subject is:

Theorem 1.0.3 (Quadratic reciprocity). *Let $p, q \geqslant 3$ be odd primes. Then*

$$(p/q)_{\mathrm{L}} \cdot (q/p)_{\mathrm{L}} = (-1)^{\frac{p-1}{2} \cdot \frac{q-1}{2}}.$$

Example 1.0.4. By quadratic reciprocity:

$$(5/13)_{\mathrm{L}} \cdot (13/5)_{\mathrm{L}} = (-1)^{\frac{5-1}{2} \cdot \frac{13-1}{2}} = (-1)^{2 \cdot 6} = 1.$$

Hence $(13/5)_{\mathrm{L}} = -1$. On the other hand,

$$(3/7)_{\mathrm{L}} \cdot (7/3)_{\mathrm{L}} = (-1)^{1 \cdot 3} = -1.$$

Hence, since $(7/3)_{\mathrm{L}} = 1$, we see that $(3/7)_{\mathrm{L}} = -1$. Actually one can restate quadratic reciprocity in the following manner:

$$(p/q)_{\mathrm{L}} = \epsilon (q/p)_{\mathrm{L}}$$

where

$$\epsilon = \begin{cases} 1 & p \text{ or } q \text{ is congruent to 1 modulo 4} \\ -1 & \text{both } p \text{ and } q \text{ are congruent to 3 modulo 4.} \end{cases}$$

There are many proofs of quadratic reciprocity. One of them, due to Tate who attributes it to Gauss, deduces it from the following isomorphism of K-groups [Mil71, Theorem 11.6]:

Theorem 1.0.5 (Tate). *There is a canonical isomorphism:*

$$\mathrm{K}_2^{\mathrm{M}}(\mathbb{Q}) \xrightarrow{\cong} \mathbb{Z}/2 \oplus \bigoplus_{p \geqslant 3} \mathbb{F}_p^{\times}.$$

I want to explain what $\mathrm{K}_2^{\mathrm{M}}$ is and what the maps are as some basic examples of motivic objects.

1.1. Milnor K-theory and symbols.

Milnor K-groups will feature heavily throughout this course; however it does seem rather mysterious why they are relevant at all. The goal of this section is to give a geometric interpretation of these symbols and explain their "motivic" nature.

Definition 1.1.1 (Symbol). Let A be a ring and let $\mathrm{A}^{\times} \subset \mathrm{A}$ the group of invertible elements of A. A **symbol**, valued in an abelian group M is a map

$$\{-,-\} : \mathrm{A}^{\times} \times \mathrm{A}^{\times} \to \mathrm{M}$$

such that

(1) $\{aa', b\} = \{a, b\} + \{a', b\}$;
(2) $\{a, bb'\} = \{a, b\} + \{a, b'\}$;
(3) $\{a, 1-a\} = 0$ if $a, 1-a \in \mathrm{A}^{\times}$.

We call (3) the **Steinberg relations**. Sometimes, for clarity, these are called **Steinberg symbols**.

Remark 1.1.2. Most textbooks define symbols only for $\mathrm{A} = \mathrm{F}$ a field. The definition of symbols and Milnor K-groups are not quite correct whenever A is an arbitrary ring, but sometimes it is. Perhaps it is best to call them "naive" symbols.

We begin with some examples of symbols.

Example 1.1.3 (The Hilbert symbol at infinity). Consider

$$c_{\infty} : \mathbb{R}^{\times} \times \mathbb{R}^{\times} \to \{\pm 1\}$$

given by

$$c_{\infty}(x, y) = \begin{cases} -1 & x, y < 0 \\ +1 & \text{else.} \end{cases}$$

Now, $c_{\infty}(x, 1-x) = 1$ must be true because x and $1-x$ cannot be negative at the same time. This is the easiest example of a symbol.

Example 1.1.4 (The tame symbol)**.** Let F be a discrete valuation field, with value group ν and κ the residue field. The **tame symbol** is given by

$$\mathrm{F}^\times \times \mathrm{F}^\times \xrightarrow{\partial_v} \kappa^\times$$

given by

$$\partial_v(a,b) = (-1)^{\nu(a)\nu(b)}\overline{\left(\tfrac{b^{\nu(a)}}{a^{\nu(b)}}\right)}$$

Let's provide a quick proof that the tame symbol satisfies the Steinberg relations. Let R be the valuation ring of F. We can write

$$a = a_1\pi^{\nu_a} \qquad b = b_1\pi^{\nu_b} \qquad a_1, b_1 \in \mathrm{R}^\times, \nu_a, \nu_b \in \mathbb{Z}.$$

Let $a+b=1$ and our goal is to prove that $\partial_v(a,b) = 0 \mod \mathfrak{m}$.

If $\nu_a > 0$, then $a \in \mathfrak{m}$. Since $b = 1-a$, it must then be a unit in R so that $\nu_b = 0$. We then calculate

$$\partial_v(a,b) = (-1)^0\overline{\left(\tfrac{b^{\nu(a)}}{a^{\nu(b)}}\right)} = \overline{b^{\nu(a)}} = \overline{b}^{\nu_a} = (1-a)^{\nu_a} \equiv 1 \mod \mathfrak{m}.$$

Evidently, if $\nu_b > 0$, the same argument works. If $\nu_a = \nu_b = 0$, then we have taken zero powers of a and b and so we get 1.

Now, assume that $\nu_a < 0$, so that $a^{-1} \in \mathfrak{m}$. We calculate

$$\tfrac{b}{a} = \tfrac{1-a}{a} = -1 + a^{-1} \equiv -1 \mod \mathfrak{m}.$$

Having $\nu_a < 0$ also means that $\nu(1-a) = \min\{\nu(-1) = 0, \nu(a)\} = \nu(a)$ and thus the term $\overline{\left(\tfrac{b^{\nu(a)}}{a^{\nu(b)}}\right)}$ evaluates as

$$\overline{\tfrac{b}{a}}^{\nu(a)} = (-1)^{\nu_a},$$

and the sign in front of the tame symbol is given by $(-1)^{\nu(1-a)\nu(a)} = (-1)^{\nu(a)}$ and thus

$$\partial_v(a,b) = (-1)^{\nu(a)}(-1)^{\nu(a)} \equiv 1 \mod \mathfrak{m}$$

as desired. The same argument also works for $\nu_b < 0$.

As suggested by the notation, the tame symbol is a kind of a boundary map for K-groups. We will use this later.

Example 1.1.5 (The differential symbol)**.** The following is the most important character in this whole story, or at least this class. Let R be a ring; we denote by $\Omega^n_{\mathrm{R}} := \bigwedge^n_{\mathbb{Z}} \Omega^1_{\mathrm{R}/\mathbb{Z}}$, the n-th wedge powers of *absolute* differential forms; I remark that this is not necessarily a good thing to look at for a arbitrary R. For $f \in \mathrm{R}^\times$ we write

$$\mathrm{dlog}(f) := \tfrac{df}{f} \in \Omega^1_{\mathrm{R}/\mathbb{Z}}.$$

The **differential symbol** is given by

$$\mathrm{R}^\times \times \mathrm{R}^\times \to \Omega^2_{\mathrm{F}} \qquad (f,g) \mapsto \mathrm{dlog}(f) \wedge \mathrm{dlog}(g).$$

We verify that

$$\begin{aligned}\mathrm{dlog}(f) \wedge \mathrm{dlog}(1-f) &= \tfrac{1}{f \cdot 1-f} df \wedge d(1-f) \\ &= -\tfrac{1}{f \cdot 1-f} df \wedge df \\ &= 0.\end{aligned}$$

Example 1.1.6 (The Galois symbol)**.** Let $\frac{1}{n} \in \mathrm{F}$ where F is a field. Then the Kummer sequence on $\mathrm{R}_{\text{ét}}$ yields a boundary map on étale cohomology

$$\mathrm{F}^\times \xrightarrow{\delta} \mathrm{H}^1_{\text{ét}}(\mathrm{F}; \mu_n).$$

The **Galois symbol** is a map

$$\mathrm{F}^\times \times \mathrm{F}^\times \to \mathrm{H}^2_{\text{ét}}(\mathrm{F}; \mu_n^{\otimes 2}) \qquad (f,g) \mapsto \delta(f) \cup \delta(g).$$

The verification of the Steinberg relation is due to Tate; one can refer to [Wei13, Proposition III.6.10.3]. For K-theory with coefficients away invertible in the base field, this is perhaps the most important symbol.

We can now write down the universal receptacle for symbols:

Definition 1.1.7. The **symbolic** K_2**-group** or **(naive) Milnor** K_2**-group** of A is defined to be the

$$\mathrm{K}_2^{\mathrm{M}}(\mathrm{A}) := \mathrm{A}^\times \otimes_{\mathbb{Z}} \mathrm{A}^\times/(a \otimes 1-a, a \neq 0,1).$$

The equivalence class of the pure tensor $x \otimes y$ will be denoted by $\{x,y\} \in \mathrm{K}_2^{\mathrm{M}}(\mathrm{A})$. In general, a typical elemeent of $\mathrm{K}_2^{\mathrm{M}}(\mathrm{A})$ is a finite linear combination of terms that look like $\{x_1,y_1\} + \cdots + \{x_n,y_n\}$. Here are a couple more relations in Milnor K-theory which are useful. In fact they are expected for a good theory of "symbolic K-groups"; see [Ker09, Lemma 2.2].

Lemma 1.1.8. *Let* B *be the localization of a local ring* A *such that* A *has infinite residue fields. Then in* $\mathrm{K}_2^{\mathrm{M}}(\mathrm{B})$ *the following hold:*

(1) $\{x,-x\} = 0$;
(2) *we have skew symmetry:* $\{x,y\} = -\{y,x\}$*; in particular* $\{x,x\} = 0$.

We remark that the above hypotheses include the case that A itself is a field.

Proof. We claim that the second relation follows from the first. We expand, via bilinearity,

$$\{xy,-xy\} = \{x,-x\} + \{x,y\} + \{y,x\} + \{y,-y\}.$$

Therefore, if we assumed (1), we get that $\{x,y\} + \{y,x\} = 0$ and hence the result is proved.

We now prove the first relation. We begin by establishing these relations for elements in the image of the induced map $\mathrm{A} \to \mathrm{B}$. First, notice that bilinearity easily implies that $-\{x,y\} = \{x^{-1},y\}$. If $x \in \mathrm{A}^\times$ such that $1-x$ is also a unit, then we can write

$$-x = \tfrac{1-x}{1-\frac{1}{x}}. \tag{1.1.9}$$

Therefore, we see that $1-\frac{1}{x}$ is in $\mathrm{A}^\times$ and that:

$$\{x,-x\} = \{x, \tfrac{1-x}{1-\frac{1}{x}}\} = \{x,1-x\} - \{x,1-\tfrac{1}{x}\} = 0 + \{\tfrac{1}{x},1-\tfrac{1}{x}\} = 0.$$

This finishes the proof whenever A is a field. If A is not a field, we have to contend with the case that $1-x \notin \mathrm{A}^\times$.

Let $s \in \mathrm{A}^\times$ such that $\overline{s} \neq 1$, which is exactly saying that $1-s \in \mathrm{A}^\times$. Then, observe that $1-xs$ is also in $\mathrm{A}^\times$ and we have:

$$0 = \{xs,-xs\} = \{x,-x\} + \{s,-s\} + \{x,s\} + \{s,x\} = \{x,-x\} + \{x,s\} + \{s,x\},$$

where we have used that $\{xs,-xs\} = \{s,s\} = 0$ from the previous case. Therefore:

$$\{x,-x\} = -\{x,s\} - \{s,x\}. \tag{1.1.10}$$

Choosing elements s_1,s_2 such that $\overline{s}_1, \overline{s}_2, \overline{s_1s_2} \neq 1$ (we can do this because of the assumption on residue fields) we have, by the previous (1.1.10) on s_1s_2

$$\{x,-x\} = -\{x,s_1s_2\} - \{s_1s_2,x\} = -\{x,s_1\} - \{x,s_2\} - \{s_1,x\} - \{s_2,x\}.$$

Using again (1.1.10) on s_1 and s_2 we conclude that the above is equal to $2\{x,-x\}$ and hence $\{x,-x\} = 0$.

Next, assume that $x \in \mathfrak{m}$ but becomes invertible in B. Then $1-x \in \mathrm{A}^\times$. Furthermore, by (1.1.9), $1-x^{-1}$ is in $\mathrm{B}^\times$ and that

$$\{x,-x\} = \{x, \tfrac{1-x}{1-\frac{1}{x}}\} = \{x,1-x\} - \{x,1-\tfrac{1}{x}\} = 0 + \{\tfrac{1}{x},1-\tfrac{1}{x}\} = 0.$$

Lastly, we prove the general case. Let $x = a/b$ where $a,b \in \mathrm{A}$ and $a,b \in \mathrm{B}^\times$. Then

$$\{a/b,-a/b\} = \{a,-a\} + \{b,b\} - \{b,-a\} - \{a,b\}.$$

The term $\{a, -a\}$ is zero by what we have already proved. On the other hand,

$$\{b, b\} = \{b, (-1)(-b)\} = \{b, -1\} + \{b, -b\} = \{b, -1\}$$

since $\{b, -b\} = 0$ by what we have already proved. Furthermore we have also already proved skew symmetry for terms which are not fractional hence

$$\{a, b\} = -\{b, a\}$$

Therefore

$$\{a/b, -a/b\} = \{b, -1\} - \{b, -a\} - \{a, b\} = \{b, -1\} - \{b, -a\} + \{b, a\},$$

which, by bilinearity, works out as

$$\{b, -1 \cdot (-a)^{-1}\} + \{b, a\} = \{b, a^{-1}\} + \{b, a\} = 0$$

□

Remark 1.1.11 (Naive versus improved Milnor K-theory)**.** In the proof of Lemma 1.1.8, it is important that we can choose certain units in the residue field and hence, we are restricted to the range that they are large enough to make these choices. This is one of the many reasons why the definition of Milnor K-theory is not quite correct. Gabber and Kerz have proposed "improved" Milnor K-theory which is the correct verson of this theory for semilocal rings.

In this manner, we can easily calculate K_2 of finite fields and prove that there is no nontrivial symbols over finite fields.

Lemma 1.1.12. $\mathrm{K}_2^{\mathrm{M}}(\mathbb{F}_q) = 0$.

Proof. Recall that $\mathbb{F}_q^\times$ is cyclic and let v be a generator. By Lemma 1.1.8, $\{v, v\} = 0$. But any element in $\mathrm{K}_2^{\mathrm{M}}(\mathbb{F}_q)$ is a linear combination of elements of the form $\{v^i, v^j\} = ij\{v, v\} = 0$.

□

Let us now sketch how Theorem 1.0.5 works and how it is related to quadratic reciprocity. Whatever K-theory is, Quillen has proved a localization sequence on the level of spectra

$$\bigoplus \mathrm{K}(\mathbb{F}_p) \xrightarrow{i_*} \mathrm{K}(\mathbb{Z}) \xrightarrow{j^*} \mathrm{K}(\mathbb{Q}).$$

We remark that the second map comes from the pullback functoriality of K-theory and the first comes from the pushforward functoriality of K-theory. Hence we have a long exact sequence [Wei13, VI.(6.6)]

$$\cdots \to \bigoplus \mathrm{K}_2(\mathbb{F}_p) \to \mathrm{K}_2(\mathbb{Z}) \to \mathrm{K}_2(\mathbb{Q}) \xrightarrow{\partial} \bigoplus \mathrm{K}_1(\mathbb{F}_p) = \mathbb{F}_p^\times \to \mathrm{K}_1(\mathbb{Z}) \cdots .$$

We have seen that $\mathrm{K}_2(\mathbb{F}_p) = 0$. Another check shows that the map labeled from $\bigoplus \mathrm{K}_1(\mathbb{F}_p) \to \mathrm{K}_1(\mathbb{Z})$ is actually zero [Wei13, Application 6.5.1]. Hence we have a short exact sequence [Wei13, Theorem III.6.5]

$$0 \to \mathrm{K}_2(\mathbb{Z}) \to \mathrm{K}_2(\mathbb{Q}) \xrightarrow{\partial} \bigoplus \mathrm{K}_1(\mathbb{F}_p) = \mathbb{F}_p^\times \to 0. \tag{1.1.13}$$

Milnor has computed $\mathrm{K}_2(\mathbb{Z}) = \{\pm 1\}$ [Mil71, Chapter 9] and, we can split the map $\mathrm{K}_2(\mathbb{Z}) \to \mathrm{K}_2(\mathbb{Q})$ using the Hilbert symbol at infinity, Example 1.1.3. Therefore we have an isomorphism implemented by the map:

$$\mathrm{K}_2(\mathbb{Q}) \xrightarrow{(c_\infty, \partial_3, \partial 5, \cdots)} \{\pm 1\} \oplus \bigoplus_{p \geqslant 3} \mathbb{F}_p^\times.$$

To prove quadratic reciprocity, we apply this isomorphism and a modified version of the even tame symbol.

Example 1.1.14 (Modified even tame symbol)**.** Any nonzero rational number can be written as a product

$$r = \pm 2^j 5^k u \qquad k = 0, 1 \qquad u = m/n, m \equiv 1 \mod 8.$$

Then write

$$x = (-1)^i_x 2^j_x 5^k_x u \qquad y = (-1)^i_y 2^{j_y} 5^{k_y} u'.$$

and set

$$\partial'_2(x, y) = (-1)^{i_x i_y + j_x k_y + k_x j_y}.$$

This is a Steinberg symbol.

Reformulating Milnor's calculation, we learn that for any other symbol c on $\mathbb{Q}$ taking values in M, we must have maps $\varphi_p : \mathbb{F}_p^\times \to \mathrm{M}$ for $p \geqslant 3$ and $\varphi_{\mathbb{Z}} : \{\pm 1\} \to \mathrm{M}$ such that

$$c(x, y) = \varphi_{\mathbb{Z}}((x, y)) \cdot \prod_{p \geqslant 3} \varphi_p(\partial_p(x, y)) \in \mathrm{M}$$

Let $\mathrm{M} = \{\pm 1\}$. The modified tame symbol is not accounted for in the classification of symbols over $\mathbb{Q}$. Hence we have must a "product formula":

$$\partial'_2(x, y) = \varphi_{\mathbb{Z}}((x, y)_\infty) \cdot \prod_{p \geqslant 3} \varphi_p(\partial_p(x, y)). \tag{1.1.15}$$

One then needs to prove that Legendre symbols do indeed belong this universe.

Lemma 1.1.16. *We have:* $\varphi_{\mathbb{Z}} = \mathrm{id}$ *and* $\varphi_p(\partial_p(x, y)) = \partial_p(x, y)^{\frac{p-1}{2}} \mod p$.

Therefore, we conclude the following refinement of (1.1.15):

$$\partial'_2(x, y) = (x, y)_\infty \cdot (\partial_p(x, y))^{\frac{p-1}{2}} \in \{\pm 1\}. \tag{1.1.17}$$

Remark 1.1.18 (Weil Reciprocity)**.** There is another reciprocity law which is the function field analog of quadratic reciprocity. Let F be a field and $\mathfrak{p}$ be a maximal ideal of $\mathrm{F}[t]$. This should be thought of as a closed point of $\mathbb{A}^1_{\mathrm{F}}$ and thus defines a discrete valuation on $\mathrm{F}(t)$. Therefore, for each such $\mathfrak{p}$, we have a tame symbol

$$\partial_{\mathfrak{p}} : \mathrm{K}_2^{\mathrm{M}}(\mathrm{F}(t)) \to (\mathrm{F}[t]/\mathfrak{p})^\times.$$

There is also the discrete valuation at infinity on $\mathrm{F}(t)$ such that for a polynomial $f \in \mathrm{F}[t]$ we have that $\nu_\infty(f) = -\deg(f)$. We also get a tame symbol at ∞:

$$\partial_\infty : \mathrm{K}_2^{\mathrm{M}}(\mathrm{F}(t)) \to \mathrm{F}^\times.$$

Weil proved

$$(f, g)_\infty \cdot \prod_{\mathfrak{p}} \mathrm{N}_{\kappa(\mathfrak{p})/\mathrm{F}}(\partial_p\{f, g\}) = 1 \in \mathrm{F}^\times$$

Here, $\mathrm{N}_{\kappa(\mathfrak{p})/\mathrm{F}} : (\mathrm{F}[t]/\mathfrak{p})^\times \to \mathrm{F}^\times$ is the field norm. This equation should be thought of as the analog of the "product formula" of (1.1.17) and is a consequence of the following analog of (1.1.13):

$$0 \to \mathrm{K}_2(\mathrm{F}) \to \mathrm{K}_2(\mathrm{F}(t)) \to \bigoplus_{\mathfrak{p} \in |\mathbb{A}^1_{\mathrm{F}}|} (\mathrm{F}[t]/\mathfrak{p})^\times \to 0.$$

There is a very concrete interpretation of Weil reciprocity that is very much analogous to quadratic reciprocity. Assume that F is algebraically closed (so that we do not need to invoke field norms). Let f, g be rational functions on $\mathbb{P}^1$ such that the supports of $\mathrm{div}(f)$ and $\mathrm{div}(g)$ are disjoint. That is to say, any factor with nonzero coefficient in $\mathrm{div}(f)$ must be zero in $\mathrm{div}(g)$ and vice versa. Therefore we can calculate the tame symbol for any valuation determined by $x \in |\mathbb{P}^1_{\mathrm{F}}|$ as

$$\partial_{\nu_x}\{f, g\} = \begin{cases} g(x)^{\nu(f)} & x \text{ is a support of } \mathrm{div}(f) \\ f(x)^{-\nu(g)} & x \text{ is a support of } \mathrm{div}(g) \\ 1 & \text{else.} \end{cases}$$

Plugging this into Weil reciprocity we get

$$\prod_{x\in \mathrm{supp}(\mathrm{div}(f))} g(x)^{\nu(f)} = \prod_{x\in \mathrm{supp}(\mathrm{div}(g))} f(x)^{\nu(g)} \in \mathrm{F}^{\times}.$$

This is the formula

$$g(\mathrm{div}(f)) = f(\mathrm{div}(g))$$

which is the nicest way to state this result.

1.2. **The motivic Steinberg relations.** We want to give a geometric explanation for the Steinberg relations. This will be an important step in naming enough interesting elements in motivic cohomology to run the proof of the Geisser-Levine theorem. It also gives a nice introduction to the "rules" of $\mathbb{A}^1$-homotopy theory. To proceed, we work over a base scheme S; we have the category $\mathrm{Sch}_{\mathrm{S}\star}$ of pointed S-schemes: these are just S-schemes X equipped with a section of the structure map $s : \mathrm{S} \to \mathrm{X}$. We have the subcategory

$$\mathrm{Sm}_{\mathrm{S}+} \subset \mathrm{Sch}_{\mathrm{S}\star}$$

of those pointed S-schemes which are of the form $\mathrm{X}_+ := \mathrm{X} \sqcup \mathrm{S}$ where X is a smooth S-scheme. There is a symmetric monoidal structure on $\mathrm{Sm}_{\mathrm{S}+}$ given by the **smash product**:

$$\mathrm{X}_+ \wedge \mathrm{Y}_+ := (\mathrm{X} \times \mathrm{Y})_+.$$

The nonabelian derived category on $\mathrm{Sm}_{\mathrm{S}+}$ is calculated as the ∞-category of pointed presheaves:

$$\mathrm{PSh}_{\Sigma}(\mathrm{Sm}_{\mathrm{S}+}) \simeq \mathrm{PSh}_{\Sigma}(\mathrm{Sm}_{\mathrm{S}})_{\star}.$$

Via the general formalism of Day convolution, $\mathrm{PSh}_{\Sigma}(\mathrm{Sm}_{\mathrm{S}})_{\star}$ acquires a symmetric monoidal structure, which we also denote by $\wedge$ such that the Yoneda functor

$$h : \mathrm{Sm}_{\mathrm{S}+} \to \mathrm{PSh}_{\Sigma}(\mathrm{Sm}_{\mathrm{S}})_{\star}$$

is symmetric monoidal and preserves sifted colimits in each variable.

Remark 1.2.1 (Calculating quotients)**.** There is one tricky aspect of $\mathrm{PSh}_{\Sigma}(\mathrm{Sch}_{\mathrm{S}})_{\star}$ which will appear in some arguments later. Let $\mathrm{W} \subset \mathrm{Y}$ be an immersion of schemes, the quotient $\mathrm{Y}/\mathrm{W} \in \mathrm{PSh}_{\Sigma}(\mathrm{Sch}_{\mathrm{S}})_{\star}$ is not calculated pointwise because taking the cofibre of the map $\mathrm{W}_+ \to \mathrm{Y}_+$ in presheaves will not give a Σ-presheaf. We need to remember that $\mathrm{PSh}_{\Sigma}(\mathrm{Sch}_{\mathrm{S}})_{\star}$ is the free ∞-category generated under sifted colimits by representables. The quotient is calculated as the pointwise geometric realization of the following diagram of presheaves of sets:

$$\mathrm{Y}_+ \leftleftarrows (\mathrm{Y} \sqcup \mathrm{W})_+ \;\substack{\leftarrow\\\leftarrow\\\leftarrow}\; (\mathrm{Y} \sqcup \mathrm{W} \sqcup \mathrm{W})_+ \;\substack{\leftarrow\\\leftarrow\\\leftarrow\\\leftarrow}\; \cdots.$$

There are several objects of interest in $\mathrm{PSh}_{\Sigma}(\mathrm{Sm}_{\mathrm{S}})_{\star}$. We have the presheaf of pointed $\mathbb{G}_m$: this is given by the cofibre of the map $h(\mathrm{S}_+) \xrightarrow{1} h(\mathbb{G}_{m+})$. We simply denoted this object by $\mathbb{G}_m$. An important object in the $\mathbb{A}^1$-invariant theory of motives are the interated smash products of $\mathbb{G}_m$, denoted by $\mathbb{G}_m^{\wedge n}$. Note that all these constructions are performed in a presheaf category and are therefore easy to calculate and describe.

So far, we have not performed any type of localization on the basic category $\mathrm{PSh}_{\Sigma}(\mathrm{Sm}_{\mathrm{S}})_{\star}$. A persistent theme in the subject is that localizations can be very hard to calculate, but also contain a very rich amount of information. Here is the first kind of localization that one encounters:

Definition 1.2.2. Let $\mathrm{X} \in \mathrm{Sch}_{\mathrm{S}}$, then consider $\mathrm{U}, \mathrm{V} \subset \mathrm{X}$ opens such that $\mathrm{U} \cup \mathrm{V} = \mathrm{X}$. We can form the following square in presheaves

$$\begin{array}{ccc} \mathrm{U} \cap \mathrm{V} & \longrightarrow & \mathrm{U} \\ \downarrow & & \downarrow \\ \mathrm{V} & \longrightarrow & \mathrm{X}; \end{array}$$

which induces the cocartesian gap $\mathrm{U} \sqcup_{\mathrm{U}\cap\mathrm{V}} \mathrm{V} \to \mathrm{X}$. The endofunctor

$$\mathrm{L}_{\mathrm{Zar}} : \mathrm{PSh}(\mathrm{Sch}_{\mathrm{S}}) \to \mathrm{PSh}(\mathrm{Sch}_{\mathrm{S}})$$

is given by localization at the maps $\mathrm{U} \sqcup_{\mathrm{U}\cap\mathrm{V}} \mathrm{V} \to \mathrm{X}$ for all $\mathrm{X}, \mathrm{U}, \mathrm{V}$ as above.

It is clear that we can consider variants of the localization above in the ∞-category of pointed presheaves or restrictions of $\mathrm{Sch}_{\mathrm{S}}$, such as to Sm_{S}; in particular the pointed version localizes at maps

$$\mathrm{U} \sqcup_{\mathrm{U}\cap\mathrm{V}} \mathrm{V}_{+} \to \mathrm{X}_{+}.$$

The other localization that we will study is **$\mathbb{A}^1$-localization**: this simply localization at the projection maps $\mathbb{A}^1_{\mathrm{X}} \to \mathrm{X}$ for all $\mathrm{X} \in \mathrm{Sch}_{\mathrm{S}}$. If L_{T} is a localization, we will say that a map $f : \mathrm{X} \to \mathrm{Y}$ is an T-equivalence if it $\mathrm{L}_{\mathrm{T}}(f)$ is an equivalence; hence there is an appropriate notion for maps to be T-contractible, T-homotopic and so on.

Theorem 1.2.3 (The motivic Steinberg relation)**.** *Let* S *be a scheme and consider the map in* $\mathrm{PSh}_{\Sigma}(\mathrm{Sm}_{\mathrm{S}+})$*:*

$$(\mathbb{A}^1 \setminus \{0,1\})_{+} \xrightarrow{\mathrm{st}} \mathbb{G}_m^{\wedge 2} \qquad a \mapsto (x, 1-x)$$

(note that this map sends $+$ *to the base point* $(1,1)$ *in* $\mathbb{G}_m \wedge \mathbb{G}_m$*. Then, this map is contractible after applying* $\mathrm{L}_{\mathrm{Zar},\mathbb{A}^1}\Sigma$*.*

As a corollary, suppose that $a \in \mathcal{O}(\mathrm{S})$ such that x and $1-x$ are both invertible. Then such a datum is classified by a map of S-schemes $\mathrm{S} \to \mathbb{A}^1 \setminus \{0,1\}_{\mathrm{S}}$; which gives rise to a map in $\mathrm{PSh}_{\Sigma}(\mathrm{Sm}_{\mathrm{X}+})$

$$\mathrm{S}^1 \simeq \Sigma(\mathrm{S}_{+}) \xrightarrow{a} \Sigma((\mathbb{A}^1 \setminus \{0,1\})_{+}).$$

The further compositon to $\mathbb{G}_m^{\wedge 2}$ is then contractible after applying $\mathrm{L}_{\mathrm{Zar},\mathbb{A}^1}$, leading to the Steinberg relation in the homotopy group

$$[\mathrm{S}^1, \mathrm{L}_{\mathrm{Zar},\mathbb{A}^1}(\mathbb{G}_m^{\wedge 2})]_{\mathrm{PSh}_{\Sigma}(\mathrm{Sm}_{\mathrm{S}})_{\star}}.$$

The following argument is due to Marc Hoyois.

Proof. Recall that the smash product is given by the cofibre sequence

$$\mathbb{G}_m \vee \mathbb{G}_m \to \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m \wedge \mathbb{G}_m.$$

Ideally, meaning if we want to prove the claim without 1-suspension, we want to factor the map

$$\mathbb{A}^1 \setminus \{0,1\} \xrightarrow{x \mapsto (x,1-x)} \mathbb{G}_m \times \mathbb{G}_m,$$

through the monomorphism $\mathbb{G}_m \vee \mathbb{G}_m$. This is not possible, but gives us a hint about how to proceed. The basic idea is to thicken the map $\mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m \wedge \mathbb{G}_m$ after 1-suspension and show that the pointed map $\mathbb{A}^1 \setminus \{0,1\}_{+} \to (\mathbb{G}_m \times \mathbb{G}_m)_{+} \to \mathbb{G}_m \wedge \mathbb{G}_m$ factors through a contractible piece of the thickening.

We set

$$\mathrm{B} := \mathrm{Bl}_{(0,1),(1,0)}(\mathbb{A}^2) \setminus \{(\mathbb{A}^1 \times 0) \cup (0 \times \mathbb{A}^1)\},$$

where the pieces removed are the strict transforms of the affine line. This is a thickening of the $\mathbb{G}_m \times \mathbb{G}_m$ where on the point $(0,1)$, we have a stuck in a copy of $\mathbb{P}^1 \setminus 0$ and similarly over the point $(1,0)$. More precisely, setting

$$\mathrm{U} := \mathrm{Bl}_{(0,1)}(\mathbb{A}^1 \times \mathbb{G}_m) \setminus (0 \times \mathbb{G}_m) \qquad \mathrm{V} := \mathrm{Bl}_{(1,0)}(\mathbb{G}_m \times \mathbb{A}^1) \setminus (\mathbb{G}_m \times 0);$$

we have an open cover of B such that $\mathrm{U} \cap \mathrm{V} = \mathbb{G}_m \times \mathbb{G}_m$. Consider C to be the closed subscheme of B given by the union of 3 affine lines: the line connecting $(1,1)$ to $(0,1)$, the exceptional divisor at $(0,1)$ and the line joining $(0,1)$ to $(1,0)$. The latter is $\mathbb{A}^1$-contractible. Hence, we have a commutative diagram in pointed presheaves (chasing the base point through to the point $(1,1)$):

$$\begin{array}{ccc} \mathbb{A}^1 \smallsetminus \{0,1\}_+ & \longrightarrow & \mathbb{G}_m \times \mathbb{G}_m \\ \downarrow & & \downarrow \\ \mathrm{C} \simeq_{\mathbb{A}^1} * & \longrightarrow & \mathrm{L}_{\mathrm{Zar}}(\mathrm{U} \sqcup_{\mathbb{G}_m \times \mathbb{G}_m} \mathrm{V}) \simeq \mathrm{B}. \end{array} \tag{1.2.4}$$

To finish the proof, we claim that the right vertical map identifies with the canonical collapse map $\mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m \wedge \mathbb{G}_m$ after $\Sigma \mathrm{L}_{\mathbb{A}^1}$.

To do so, note that we have an $\mathrm{L}_{\mathbb{A}^1}$-equivalence

$$(\mathbb{A}^1 \times 1) \sqcup_{\mathbb{G}_m \times 1} (\mathbb{G}_m \times \mathbb{G}_m) \sqcup_{1 \times \mathbb{G}_m} (1 \times \mathbb{A}^1) \simeq \mathbb{G}_m \wedge \mathbb{G}_m;$$

one sees this by noting that $(\mathbb{A}^1 \times 1) \sqcup_{\mathbb{G}_m \times 1} (\mathbb{G}_m \times \mathbb{G}_m)$ is a $\mathrm{L}_{\mathbb{A}^1}$-equivalent to coning off $\mathbb{G}_m \times 1 \subset (\mathbb{G}_m \times \mathbb{G}_m)$. Now we claim that there is an $\Sigma \mathrm{L}_{\mathbb{A}^1}$-equivalence

$$(\mathbb{A}^1 \times 1) \sqcup_{\mathbb{G}_m \times 1} (\mathbb{G}_m \times \mathbb{G}_m) \sqcup_{1 \times \mathbb{G}_m} (1 \times \mathbb{A}^1) \simeq \mathrm{U} \sqcup_{\mathbb{G}_m \times \mathbb{G}_m} \mathrm{V}.$$

We do this by producing $\Sigma \mathrm{L}_{\mathbb{A}^1}$-equivalences

$$e : (\mathbb{A}^1 \times 1) \sqcup_{\mathbb{G}_m \times 1} (\mathbb{G}_m \times \mathbb{G}_m) \to \mathrm{U} \qquad e' : (\mathbb{G}_m \times \mathbb{G}_m) \sqcup_{1 \times \mathbb{G}_m} (1 \times \mathbb{A}^1) \to \mathrm{V},$$

and then gluing them.

Consider the following diagram

$$\begin{array}{ccc} (\mathbb{A}^1 \times 1) \sqcup_{\mathbb{G}_m \times 1} (\mathbb{G}_m \times \mathbb{G}_m) & \xrightarrow{\quad e \quad} & \mathrm{U} \\ \downarrow & & \downarrow \\ (\mathbb{A}^1 \times 1) \sqcup_{\mathbb{G}_m \times 1} (\mathbb{G}_m \times \mathbb{A}^1) & \longrightarrow & \mathrm{Bl}_{(0,1)}(\mathbb{A}^2) \smallsetminus (0 \times \mathbb{A}^1). \end{array}$$

We claim that the bottom row is $\mathrm{L}_{\mathbb{A}^1}$-homotopic to a map between $\mathbb{A}^1$-contractible spaces. If this were the case, then extending vertically by taking cofibres and $\mathrm{L}_{\mathbb{A}^1}$ we get

$$\begin{array}{ccc} (\mathbb{A}^1 \times 1) \sqcup_{\mathbb{G}_m \times 1} (\mathbb{G}_m \times \mathbb{G}_m) & \xrightarrow{\quad e \quad} & \mathrm{U} \\ \downarrow & & \downarrow \\ * & \longrightarrow & * \\ \downarrow & & \downarrow \\ \frac{\mathbb{G}_m \times \mathbb{A}^1}{\mathbb{G}_m \times \mathbb{G}_m} & \xrightarrow{\quad \Sigma e \quad} & \frac{\mathrm{Bl}_{(0,1)}(\mathbb{A}^2) \smallsetminus (0 \times \mathbb{A}^1)}{\mathrm{U}}. \end{array}$$

But it is easy to see that the bottom map is an equivalence of pointed presheaves and thus $\Sigma \mathrm{L}_{\mathbb{A}^1} e$ is an equivalence. Note first that, by inspection, $(\mathbb{A}^1 \times 1) \sqcup_{\mathbb{G}_m \times 1} (\mathbb{G}_m \times \mathbb{A}^1) \cong \mathrm{Bl}_{(0,1)}(\mathbb{A}^2) \smallsetminus (0 \times \mathbb{A}^1)$. Hence it suffices to prove that $(\mathbb{A}^1 \times 1) \sqcup_{\mathbb{G}_m \times 1} (\mathbb{G}_m \times \mathbb{A}^1)$ is $\mathbb{A}^1$-contractible, we note that $\mathbb{G}_m \times 1 \to \mathbb{A}^1 \times 1$ is an $\mathrm{L}_{\mathbb{A}^1}$-equivalence, hence the map $\mathbb{A}^1 \times 1 \to (\mathbb{A}^1 \times 1) \sqcup_{\mathbb{G}_m \times 1} (\mathbb{G}_m \times \mathbb{A}^1)$ is an $\mathrm{L}_{\mathbb{A}^1}$-equivalence and the domain is $\mathbb{A}^1$-contractible. The same argument applies to $e' : (1 \times \mathbb{A}^1) \sqcup_{1 \times \mathbb{G}_m} (\mathbb{G}_m \times \mathbb{G}_m) \to \mathrm{V}$. Therefore, we see that, suspending (1.2.4) we obtain a $\mathrm{L}_{\mathbb{A}^1, \mathrm{Zar}}$-commutative diagram:

$$\begin{array}{ccc} \Sigma(\mathbb{A}^1 \smallsetminus \{0,1\})_+ & \longrightarrow & \Sigma(\mathbb{G}_m \times \mathbb{G}_m) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma \mathrm{B} \simeq \Sigma(\mathbb{G}_m \wedge \mathbb{G}_m). \end{array}$$

□

Theorem 1.2.3 hints at the naturality of Milnor K-theory — its seemingly *ad hoc* definition has a geometric explanation within the context of $\mathbb{A}^1$-homotopy theory. In the next section, we will elaborate on the category of $\mathbb{A}^1$-invariant motivic spectra over a base.

2. The formalism of $\mathbb{A}^1$-invariant motivic spectra

The Steinberg relation is one of the many reasons why $\mathbb{A}^1$-localization is a good idea. We have also seen that $\mathbb{A}^1$-localization should be coupled with Zariski localization. For the purposes of building a good theory, it turns out that one should use a topology which is finer than the Zariski topology called the Nisnevich topology. We now explain what we mean by an $\mathbb{A}^1$-invariant cohomology theory in algebraic geometry.

Definition 2.0.1. Let S be a scheme.

(1) a **Nisnevich square** is a pullback square of S-schemes

$$\begin{array}{ccc} \mathrm{U} \times_{\mathrm{X}} \mathrm{V} & \longrightarrow & \mathrm{V} \\ \downarrow & & \downarrow p \\ \mathrm{U} & \xrightarrow{j} & \mathrm{X}. \end{array}$$

where p is étale, j is open and p induces an isomorphism $p : p^{-1}((\mathrm{X} \smallsetminus \mathrm{U})_{\mathrm{red}}) \to (\mathrm{X} \smallsetminus \mathrm{U})_{\mathrm{red}}$;

(2) an **affine Nisnevich square** is a pullback square of affine S-schemes

$$\begin{array}{ccc} \operatorname{Spec} \mathrm{B}[\frac{1}{f}] & \longrightarrow & \operatorname{Spec} \mathrm{B} \\ \downarrow & & \downarrow p \\ \operatorname{Spec} \mathrm{A}[\frac{1}{f}] & \xrightarrow{j} & \operatorname{Spec} \mathrm{A}, \end{array}$$

where p is étale, $f \in \mathrm{A}$ and there is an induced isomorphsim $\mathrm{B}/f \cong \mathrm{A}/f$.

(3) A functor: $\mathrm{E} : \mathrm{Sm}_{\mathrm{B}}^{\mathrm{op}} \to \mathrm{Spt}$ is said to be an $\mathbb{A}^1$**-invariant** if the projection map $\mathrm{X} \times \mathbb{A}^1 \to \mathrm{X}$ induces an equivalence $\mathrm{E}(\mathrm{X}) \xrightarrow{\simeq} \mathrm{E}(\mathrm{X} \times \mathbb{A}^1)$.

(4) For a presheaf E, set

$$\Omega_{\mathbb{P}^1} \mathrm{E}(\mathrm{X}) := \mathrm{fib}(\mathrm{E}(\mathrm{X} \times \mathbb{P}^1) \xrightarrow{\infty^*} \mathrm{E}(\mathrm{X}));$$

which defines a functor $\Omega_{\mathbb{P}^1} \mathrm{E} : \mathrm{Sm}_{\mathrm{B}}^{\mathrm{op}} \to \mathrm{Spt}$. Since ∞^* is split by $\mathbb{P}^1 \times \mathrm{X} \to \mathrm{X}$, we have a direct sum decomposition

$$\mathrm{E}(\mathrm{X} \times \mathbb{P}^1) \simeq \mathrm{E}(\mathrm{X}) \oplus \Omega_{\mathbb{P}^1} \mathrm{E}(\mathrm{X}).$$

(5) Let $\{\mathrm{E}(\bullet)\}$ be a $\mathbb{Z}$-graded collection of presheaves of spectra, then a $\mathbb{P}^1$**-prespectrum** is a the data of maps:

$$\mathrm{E}(j) \to \Omega_{\mathbb{P}^1} \mathrm{E}(j+1).$$

(6) An $\mathbb{A}^1$**-invariant motivic cohomology theory** is the data of a $\mathbb{P}^1$-prespectrum $\{\mathrm{E}(\bullet)\}$ such that:
   (a) $\mathrm{E}(j)$ is an $\mathbb{A}^1$-invariant Nisnevich sheaf;
   (b) each $\mathbb{P}^1$-bundle datum is an equivalence.

As defined, the category of $\mathbb{A}^1$-invariant motivic cohomology theory is rather unwiedly; for example it is not quite clear that one can endow it with a symmetric monoidal structure. Nonetheless, we go ahead and denote this category by $\mathbf{SH}(\mathrm{S})$, this is the stable ∞-category of a $\mathbb{A}^1$**-invariant motivic spectra**. We give examples in the next lectures.

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