LECTURES 7-9

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We take the convention that $\mathbb{G}_m$ is the scheme $\mathbb{A}^1 \setminus 0$ pointed at 1. This can also be interpreted in the world of pointed ($\Sigma-$)presheaves. We will also denote by $\mathrm{CAlg}^{\heartsuit} \subset \mathrm{CAlg}$ the inclusion of (discrete) commutative rings into $\mathbb{E}_\infty$-rings.

1. Motivic spectra: examples and universal properties

In the previous set of lectures, we defined what it means to define an $\mathbb{A}^1$-invariant motivic cohomology theory over a base scheme S. We have alluded that they assemble into a category called **SH**(S). We start this lecture by explaining a "minimalist" approach to such cohomology theories and explain some other fundamental relations in the world of $\mathbb{A}^1$-invariant motivic homotopy theory.

1.1. **The minimalist approach.** Sometimes, we want to only contemplate cohomology theories defined on affine schemes. Let R be a fixed commutative ring which we think of as a "base." To begin with, recall the following definition due to Bass.

Definition 1.1.1 (Contraction)**.** Let $\mathcal{C}$ be a pointed ∞-category:

$$\mathrm{X} : \mathrm{SmCAlg}_{\mathrm{R}} \to \mathcal{C}.$$

Its **contraction**, is the presheaf defined as

$$\mathrm{X}_{-1}(\mathrm{S}) := \mathrm{fib}(\mathrm{X}(\mathrm{S}[t,t^{-1}]) \xrightarrow{t=1} \mathrm{X}(\mathrm{S}));$$

note that we have a canonical splitting (induced by the map $\mathrm{S} \to \mathrm{S}[t,t^{-1}]$)

$$\mathrm{X}(\mathrm{S}[t,t^{-1}]) \simeq \mathrm{X}_{-1}(\mathrm{S}) \oplus \mathrm{X}(\mathrm{S}).$$

Remark 1.1.2. If $\mathcal{C}$ is a closed symmetric monoidal, pointed ∞-category (like the ∞-category of pointed spaces or the 1-category of pointed sets), then the contraction is given by as the internal mapping object in presheaves.

$$\Omega_{\mathbb{G}_m}\mathrm{X} := \mathrm{map}(\mathbb{G}_m, \mathrm{X}) \simeq \mathrm{X}_{-1} \in \mathrm{PSh}(\mathrm{SmCAlg}_{\mathrm{R}}^{\mathrm{op}}; \mathcal{C});$$

here, as per convention, $\mathbb{G}_m$ is the representable presheaf pointed at 1. There is something which is subtle and important about contractions: suppose that τ is a topology on $\mathcal{C}$. Then the ∞-category of sheaves is stable under taking fibres, hence $(\mathrm{L}_\tau\mathrm{X})_{-1}$ is τ-sheaf. We then have a comparison map

$$\mathrm{L}_\tau(\mathrm{X}_{-1}) \to (\mathrm{L}_\tau\mathrm{X})_{-1}.$$

It is not obvious that this map is an equivalence. This type of issues will be addressed later in the course.

Example 1.1.3 (Weight shifting)**.** To preview what will happen, let us compute examples of contractions. Assume that R is a domain, then as abelian groups:

$$\mathcal{O}(\mathrm{R}[t,t^{-1}])^\times \cong \mathcal{O}(\mathrm{R})^\times \oplus \{t^n : n \in \mathbb{Z}\}.$$

Hence, thinking of $\mathcal{O}^\times$ as a presheaf of abelian groups on smooth R-algebras, we see that $\mathcal{O}^\times_{-1} \cong \mathbb{Z}$. We can also calculate the same example for Picard group. For a smooth k-algebra R where k is a field (but holds in somewhat greater generality) we have

$$\mathbb{Z} \xrightarrow{1 \mapsto [\mathrm{Spec}\,\mathrm{R} \times 0]} \mathrm{Pic}(\mathrm{R}[t]) \xrightarrow{j^*} \mathrm{Pic}(\mathrm{R}[t,t^{-1}]) \to 0.$$

The composite of the first map along with the map $\mathrm{Pic}(\mathrm{R}[t]) \xrightarrow{t=1} \mathrm{Pic}(\mathrm{R})$ is zero, but the latter map is an isomorphism by the $\mathbb{A}^1$-invariance of Picard groups in this setting. Hence we conclude that

$$\mathrm{Pic}_{-1} \simeq 0$$

as a presheaf on smooth schemes over a fixed field.

Definition 1.1.4. A **minimalist $\mathbb{A}^1$-invariant motivic cohomology theory** is the data of

(1) a collection of presheaves

$$\mathrm{X}(j) : \mathrm{SmCAlg}_{\mathrm{R}} \to \mathrm{Spt} \qquad j \in \mathbb{Z};$$

(2) bonding maps

$$\epsilon_j : \mathrm{X}(j)[j] \to (\mathrm{X}(j+1)[j+1])_{-1}$$

such that:

(1) each $\mathrm{X}(j)$ converts an affine Nisnevich square to a bicartesian diagram;
(2) each $\mathrm{X}(j)$ is a $\mathbb{A}^1$-invariant;
(3) the maps ϵ_j are equivalences.

Each $\mathbb{A}^1$-invariant motivic cohomology theory determines a minimalist one and vice versa[1]. We will not make this too precise, but the procedure is given by right Kan extension plus the following two results:

Proposition 1.1.5. *Consider the zig-zag in* $\mathrm{PSh}_\Sigma(\mathrm{Sm}_\mathrm{S})_*$*:*

$$\Sigma(\mathbb{G}_m/1) \xleftarrow{\alpha} (\mathbb{A}^1_+)/\mathbb{A}^1 \smallsetminus 0_+ \xrightarrow{\beta} \mathbb{P}^1_+/\mathbb{P}^1 \smallsetminus 0_+ \xleftarrow{\gamma} \mathbb{P}^1/\infty.$$

Then: α is an $\mathrm{L}_{\mathbb{A}^1}$-equivalence, β is a $\mathrm{L}_{\mathrm{Zar}}$-equivalence and γ is an $\mathrm{L}_{\mathbb{A}^1}$-equivalence.

Proof. The map α is induced by

$$\begin{array}{ccccc} \mathbb{A}^1 \smallsetminus 0_+ & \longrightarrow & \mathbb{A}^1_+ & \longrightarrow & (\mathbb{A}^1_+)/\mathbb{A}^1 \smallsetminus 0_+ \\ \downarrow & & \downarrow & & \downarrow \alpha \\ \mathbb{G}_m & \longrightarrow & * & \longrightarrow & \Sigma(\mathbb{G}_m/1). \end{array}$$

The left square is an $\mathrm{L}_{\mathbb{A}^1}$-cocartesian square whence α is an equivalence. For the second square we have the standard Zariski cover of $\mathbb{P}^1$:

$$\begin{array}{ccccc} \mathbb{A}^1 \smallsetminus 0_+ & \longrightarrow & \mathbb{A}^1_+ & \longrightarrow & (\mathbb{A}^1_+)/\mathbb{A}^1 \smallsetminus 0_+ \\ \downarrow & & \downarrow & & \downarrow \beta \\ \mathbb{P}^1 \smallsetminus 0_+ & \longrightarrow & \mathbb{P}^1_+ & \longrightarrow & \mathbb{P}^1_+/\mathbb{P}^1 \smallsetminus 0_+. \end{array}$$

Here, the left square is $\mathrm{L}_{\mathrm{Zar}}$-cocartesian and hence β is an equivalence. Lastly, the inclusion $\{\infty\} \hookrightarrow \mathbb{P}^1 \smallsetminus 0$ is an $\mathrm{L}_{\mathbb{A}^1}$-equivalence which explains the $\mathrm{L}_{\mathbb{A}^1}$-equivalence of γ. □

Proposition 1.1.5 is often stated as the $\mathrm{L}_{\mathrm{Zar},\mathbb{A}^1}$-equivalence

$$\Sigma(\mathbb{G}_m/1) \simeq_{\mathrm{Zar},\mathbb{A}^1} \mathbb{P}^1/\infty.$$

It implies that for any $\mathbb{A}^1$-invariant Nisnevich (in fact, Zariski) sheaf of spectra, E, we have an equivalence

$$\mathrm{E}(\mathbb{P}^1_\mathrm{X}) \simeq \mathrm{E}(\mathbb{G}_{m,\mathrm{X}})[-1],$$

which lets us freely translate between bonding maps involving $\mathbb{P}^1_\mathrm{X}$ and bonding maps involving contractions.

To translate between the affine Nisnevich and the Nisnevich topologies, we need the following result due to Asok-Hoyois-Wendt [AHW16, Proposition 2.3.2].

[1]The translation is a bit awkward: if $\mathrm{E}(j)$ is an $\mathbb{A}^1$-invariant motivic cohomology theory then its corresponding minimalist version is given by $\mathrm{X}(j) := \mathrm{E}(j)[-j]|_{\mathrm{SmCAlg}_\mathrm{R}}$.

Proposition 1.1.6. *Right Kan extension implements an equivalence*

$$\mathrm{Shv}_{\mathrm{AffNis}}(\mathrm{SmCAlg}_{\mathrm{R}}^{\mathrm{op}}) \xrightarrow{\simeq} \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_{\mathrm{R}}).$$

1.2. **Examples of motivic cohomology theories.** We now arrive at examples of motivic cohomology theories:

Example 1.2.1 (Étale cohomology)**.** Let $\frac{1}{p} \in \mathcal{O}_{\mathrm{B}}$ and $r \geqslant 1$. Then set

$$\mathrm{E}(j) := \mathrm{R}\Gamma_{\text{ét}}(-; \mu_{p^r}^{\otimes j})[2j].$$

Standard facts from étale cohomology theory tells us that $\mathrm{E}(j)$ is an $\mathbb{A}^1$-invariant Nisnevich sheaf (in fact, an étale sheaf). The theory of chern classes in étale cohomology produces, for each line bundle $\mathcal{L}$ on X, a first Chern class class $c_1(\mathcal{L}) \in \mathrm{H}^2_{\text{ét}}(\mathrm{X}; \mu_{p^r})$. From this we can produce the $\mathbb{P}^1$-bundle datum at all levels; the fact that the map

$$\mathrm{H}^i_{\text{ét}}(\mathrm{X}; \mu_{p^r}^{\otimes j}) \oplus \mathrm{H}^{i-2}_{\text{ét}}(\mathrm{X}; \mu_{p^r}^{\otimes j-1}) \xrightarrow{\pi^* \oplus \pi^* \cup c_1(\mathcal{O}(1))} \mathrm{H}^i_{\text{ét}}(\mathbb{P}^1 \times \mathrm{X}; \mu_{p^r}^{\otimes j})$$

is an equivalence (the projective bundle formula) tells us that we have a homotopy invariant motivic cohomology theory prescribed by $\mathrm{E}(\bullet) := \mathrm{R}\Gamma_{\text{ét}}(-; \mu_{p^r}^{\otimes \bullet})$. We denote this as $\mathrm{H}_{\text{ét}}\mu_{p^r,\mathrm{S}} \in$ $\mathbf{SH}(\mathrm{S})$ and suppress the subscript S whenever the context is clear.

Example 1.2.2 (de Rham cohomology in characteristic zero)**.** Let $f : \mathrm{X} \to \mathrm{S}$ be a smooth morphism[2] then form a chain complex of $\mathcal{O}_{\mathrm{X}}$-modules:

$$\Omega^\bullet_{\mathrm{X/S}} = [0 \to \mathcal{O}_{\mathrm{X}} \xrightarrow{d} \Omega^1_{\mathrm{X/S}} \to \Omega^2_{\mathrm{X/S}} \to \cdots \Omega^q_{\mathrm{X/S}} \to],$$

called the **relative de Rham complex**. The **(relative) de Rham cohomology** of f is the presheaf on Sm_{S} given by

$$\mathrm{dR}_{-/\mathrm{S}} : (\mathrm{X} \to \mathrm{S}) \mapsto \mathrm{R}\Gamma(\mathrm{X}, \Omega^\bullet_{\mathrm{X/S}}) \in \mathrm{D}(\mathrm{S}).$$

If we set

$$\mathrm{E}(j) := \mathrm{dR}_{-/\mathrm{S}}[2j]$$

then we almost get an example. Indeed, by virtue of being an instance of coherent cohomology, $\mathrm{dR}_{-/\mathrm{S}}$ is an étale sheaf on Sm_{S} and hence a Nisnevich sheaf. Furthermore, we have a chern class map on de Rham cohomology given by [Stacks, Tag 0FLE]:

$$\mathrm{Pic}(\mathrm{X}) \to \mathrm{H}^2(\mathrm{dR}_{\mathrm{X/S}}).$$

This gives rise to an equivalence [Stacks, Tag 0FMS]:

$$\mathrm{dR}_{\mathrm{X/S}} \xrightarrow{\pi^* \oplus \pi^* \cup c_1(\mathcal{O}(1))} \mathrm{dR}_{\mathbb{P}^1_{\mathrm{X}}/\mathrm{S}} \oplus \mathrm{dR}_{\mathbb{P}^1_{\mathrm{X}}/\mathrm{S}}[2].$$

is an equivalence. However, de Rham cohomology fails to be $\mathbb{A}^1$-invariant; it is only so in characteristic zero [Stacks, Tag 0FUI].

So let us restrict ourselves in this setting. We can describe the bonding map in terms of a miminalist $\mathbb{A}^1$-invariant motivic cohomology theory as follows:

$$\mathrm{dR}_{\mathrm{X/S}} \to \mathrm{dR}_{\mathbb{G}_m,\mathrm{X}/\mathrm{S}}[1] (\simeq \mathrm{dR}_{\mathbb{P}^1_{\mathrm{X}}/\mathrm{S}}[2]) \qquad \omega \mapsto \tfrac{dt}{t} \wedge \omega.$$

Then, setting

$$\mathrm{dR}(j) := \mathrm{dR}_{-/\mathrm{S}}[j],$$

we obtain an minimalist $\mathbb{A}^1$-invariant motivic cohomoology theory which we denote by $\mathrm{HdR}_{\mathrm{S}}$.

[2]What follows can be defined more generally, but are usually pathological; what one needs to do instead is to *animate*, a technique which we will find invaluable in the course of our adventure.

1.3. **Algebraic K-theory.** In this lecture, we begin properly discussing and constructing algebraic K-theory; however we will not give a comprehensive treatement of this object — that is for another class. We begin with the most elementary possible definition. First, let us recall the theory of $\mathbb{E}_\infty$-spaces; such an object is a functor

$$\mathrm{M} : \mathrm{Fin}_\star \to \mathrm{Spc},$$

satisfying the **Segal condition**: for each $n \geqslant 1$ we have maps

$$\rho_i : \{1, \cdots, n\}_+ \to \{1\}_+ \qquad j \neq i \mapsto +, i \mapsto 1,$$

which altogether induce

$$\prod \mathrm{M}(\rho_i) : \mathrm{M}(\{1, \cdots, n\}_+) \to \mathrm{M}(\{1\}_+);$$

and we ask that these maps are equivalences and that the canonical map

$$* \to \mathrm{M}(\emptyset_+)$$

is as well. This forms a subcategory $\mathrm{CMon} \subset \mathrm{Fun}(\mathrm{Fin}_\star, \mathrm{Spc})$, the higher algebra version of the theory of commutative monoids. Objects in this subcategory are called $\mathbb{E}_\infty$-spaces: roughly they are spaces equipped with multiplications which are coherently commutative.

The map

$$\{1, 2\}_+ \to \{1\}_+ \qquad 1, 2 \mapsto 1.$$

induces a "multiplication" (up to a choice of a homotopy inverse of the map labeled an equivalence)

$$\mathrm{M}(\{1\}_+) \times \mathrm{M}(\{1\}_+) \xleftarrow{\simeq} \mathrm{M}(\{1, 2\}_+) \to \mathrm{M}(\{1\}_+);$$

and since π_0 preserves products, $\pi_0(\mathrm{M}(\{1\}_+))$ has the structure of a classical commutative monoid. An $\mathbb{E}_\infty$-space is said to be **grouplike** if $\pi_0(\mathrm{M}(\{1\}_+))$ is a group. The inclusion of grouplike $\mathbb{E}_\infty$-spaces:

$$\mathrm{CGrp} \subset \mathrm{CMon}$$

admits a left adjoint $\mathrm{M} \mapsto \mathrm{M}^{\mathrm{grp}}$ called **group completion**. The latter procedure is somewhat subtle: while it makes π_0 of M commutative, it does so by introducing higher homotopy groups into M.

Remark 1.3.1 (The group completion theorem)**.** While the homotopy groups of $\mathrm{M}^{\mathrm{grp}}$ are complicated, a theorem of McDuff and Segal shows that the homology of $\mathrm{M}^{\mathrm{grp}}$ is not too bad. To state their result, note that if M is an H-space, then $\mathrm{H}_*(\mathrm{M})$ admits the structure of a graded ring. Elements of $\pi_0(\mathrm{M})$ then form a multiplicative subset under the map $\pi_0(\mathrm{M}) \to \mathrm{H}_0(\mathrm{M}) = \mathbb{Z}[\pi_0(\mathrm{M})]$ whence we can look at the localization $\mathrm{H}_*(\mathrm{M})[\pi_0(\mathrm{M})^{-1}]$.

Theorem 1.3.2 (McDuff-Segal)**.** *The map* $\mathrm{H}_*(\mathrm{M}) \to \mathrm{H}_*(\mathrm{M}^{\mathrm{grp}})$ *witnesses a localization at* $\pi_0(\mathrm{M})^{-1}$*. In particular we have an isomorphism of graded rings*

$$\mathrm{H}_*(\mathrm{M}^{\mathrm{grp}}) \simeq \mathrm{H}_*(\mathrm{M})[\pi_0(\mathrm{M})^{-1}].$$

Remark 1.3.3 (Refinements by Nikolaus)**.** Let $\mathrm{M} \in \mathrm{CMon}$, $m \in \pi_0(\mathrm{M})$. We set

$$\mathrm{T}_m\mathrm{M} := \mathrm{colim}(\mathrm{M} \xrightarrow{+m} \mathrm{M} \xrightarrow{+m} \mathrm{M} \xrightarrow{+m} \cdots);$$

this colimit is taken in spaces, but we see that it can be calculated in the ∞-category of M-modules. This means that the resulting object inherits the natural structure of an M-module. Let $\{m_1, \cdots, m_n\}$ be an ordered set of elements of $\pi_0(\mathrm{M})$, then we can inductively define

$$\mathrm{T}_{\{m_1, \cdots, m_n\}}\mathrm{M} := \mathrm{T}_{m_n}(\mathrm{T}_{\{m_1, \cdots, m_{n-1}\}}\mathrm{M}).$$

Let us write

$$\mathrm{T}_\infty\mathrm{M} := \operatorname*{colim}_{\mathrm{S} \subset \mathrm{I}} \mathrm{T}_{\mathrm{S}}\mathrm{M},$$

where $\{m_i\} \subset \pi_0(\mathrm{M})$ are generators of the monoid, that we give some ordering. We have map

$$\mathrm{T}_\infty\mathrm{M} \to \mathrm{T}_\infty\mathrm{M}^{\mathrm{grp}} \simeq \mathrm{M}^{\mathrm{grp}}, \tag{1.3.4}$$

where the equivalence is because the operation on a group does not do anything since action by any element of $\pi_0(\mathrm{M})$ acts invertibly on M. Nikolaus has proved the following refienment of the group completion theorem: the map (1.3.4) is an equivalence if and only if $\pi_1(\mathrm{T}_\infty\mathrm{M}, x)$ is abelian for any $x \in \pi_0(\mathrm{T}_\infty\mathrm{M}, x) = \pi_0(\mathrm{M})^{\mathrm{grp}}$. An explanation is in the appendix.

Remark 1.3.5 (Monoids in the homotopy category)**.** Any $\mathrm{X} \in \mathrm{CMon}$ defines an object $\mathrm{X} \in \mathrm{CMon}(\mathrm{hSpc})$, a homotopy type equipped with a multiplication which is commutative, unital and associative up to (incoherent) homotopies. This is an example of a **H-space**, a pointed space $e : * \to \mathrm{H}$ equipped with a pairing

$$\mu : \mathrm{H} \times \mathrm{H} \to \mathrm{H}$$

which is only homotopy unital: $\mu(x, e) \simeq \mu(e, x) \simeq x$. This is enough to run the Eckman-Hilton agument to conclude that $\pi_1(\mathrm{H}, e)$ is an abelian group.

This procedure leads us to the construction of:

Construction 1.3.6 (Connective K-theory of a ring)**.** Let R be a commutative ring. We let $\mathrm{Proj}^{\mathrm{fg}}(\mathrm{R})$ be the 1-category of finitely generated projective R-modules. Under the operation of direct sum, $\mathrm{Proj}^{\mathrm{fg}}(\mathrm{R})$ is a symmetric monoidal category and thus taking the maximal subgroupoid we get $\iota\mathrm{Proj}^{\mathrm{fg}}(\mathrm{R})$, a 1-groupoid which we can regard as a space; by the symmetric monoidality of $\oplus$, $\iota\mathrm{Proj}^{\mathrm{fg}}(\mathrm{R})$ inherits the structure of an $\mathbb{E}_\infty$-space from $\oplus$. Set the **connective K-theory of** R to be

$$\mathrm{K}_{\geqslant 0}(\mathrm{R}) := (\iota\mathrm{Proj}^{\mathrm{fg}}(\mathrm{R}))^{\mathrm{grp}},$$

the group completion. The K-groups of R is then given by:

$$\mathrm{K}_0(\mathrm{R}) = \pi_0(\mathrm{K}(\mathrm{R}));$$
$$\mathrm{K}_j(\mathrm{R}) := \pi_j(\mathrm{K}_{\geqslant 0}(\mathrm{R}), \mathcal{O}) \qquad j \geqslant 1.$$

Remark 1.3.7 (K-theory of more general rings)**.** From Construction 1.3.6, it is evident that K-theory of more general rings can be easily defined: for any $\mathbb{E}_1$-ring R, one considers $\mathrm{Perf}_{\mathrm{R}} \subset \mathrm{RMod}_{\mathrm{R}}$ the smallest stable subcategory of the ∞-category of right R-modules which contains R and closed under retracts; this is the higher algebra version of finitely generated projective modules. Then $\mathrm{K}(\mathrm{R}) := (\iota\mathrm{Perf}_{\mathrm{R}})^{\mathrm{grp}}$.

Example 1.3.8 (K_0 of a ring)**.** By construction, we know exactly what K_0 is: it is given by the group completion of the commutative monoid of isomorphism classes of finitely generated projective modules.

Remark 1.3.9 (Ring structure)**.** We can assemble K-theory into a graded ring

$$\mathrm{K}_*(\mathrm{R}) := \bigoplus_{j \geqslant 0} \mathrm{K}_j(\mathrm{R}),$$

where the product is a shadow of the fact that $\mathrm{K}_{\geqslant 0}(\mathrm{R})$ has an additional $\mathbb{E}_\infty$-ring structure induced by the $\otimes$-structure on finitely generated projective modules.

Remark 1.3.10 (K-theory spectrum)**.** It is convenient to regard connective K-theory as a spectrum, in view of a globalization procedure that we will soon perform. We write CMon the ∞-category of commutative monoids in anima, i.e., the ∞-category of $\mathbb{E}_\infty$-spaces. Classically, we have an equivalence between connective spectra and grouplike commutative monoids:

$$\mathrm{Spt}_{\geqslant 0} \simeq \mathrm{CGrp}.$$

To be slightly more explicit about this equivalence: we have the following diagram

$$\begin{array}{ccc} \mathrm{Spt}_{\geqslant 0} & \hookrightarrow & \mathrm{Spt} \\ \simeq\big\downarrow & \swarrow & \big\downarrow\Omega^\infty \\ \mathrm{CGrp} & \xrightarrow{\text{forget}} & \mathrm{Spc}. \end{array}$$

Hence, the equivalence between connective spectra and grouplike commutative monoids above is implemented by taking the infinite loop space. The product structure from the previous discussion then says that $K_{\geqslant 0}(R)$ is a $\mathbb{E}_\infty$-ring spectrum. Without further comment, we consider connective K-theory as a presheaf of spectra:

$$K_{\geqslant 0} : \mathrm{CAlg}^{\heartsuit} \to \mathrm{CAlg}.$$

Example 1.3.11 (The determinant and rank)**.** We revisit the constructions of determinant and rank at this level. Let $\mathcal{P}\mathrm{ic}(R) \subset \iota\mathrm{Proj}^{\mathrm{fg}}(R)$ be the subgroupoid of invertible R-modules, i.e., line bundles on $\mathrm{Spec}\, R$. Under tensor product, $\mathcal{P}\mathrm{ic}(R) \in \mathrm{CGrp}$. It has homotopy groups given by:

$$\pi_0(\mathcal{P}\mathrm{ic}(R)) = \mathrm{Pic}(R) \qquad \pi_1(\mathcal{P}\mathrm{ic}(R), \mathcal{O}) \cong R^\times.$$

We have a morphism between presheaves of groupoids on commutative rings:

$$\det : \iota\mathrm{Proj}^{\mathrm{fg}} \to \mathcal{P}\mathrm{ic}$$

which sends a module M to its top exterior power; this morphism preserves the base point $\mathcal{O}$ and converts $\oplus$ to $\otimes$. We want to say that this map is a morphism of $\mathbb{E}_\infty$-spaces but this is *not* the case: the diagram of isomorphisms:

$$\begin{array}{ccc} \det(M \oplus N) & \xrightarrow{\simeq} & \det(M) \otimes \det(N) \\ \simeq \downarrow & & \downarrow \simeq \\ \det(N \oplus M) & \xrightarrow{\simeq} & \det(N) \otimes \det(M) \end{array}$$

only commutes up to a sign given by $(-1)^{\mathrm{rank}(M)\mathrm{rank}(N)}$. We thus only have an induced map $K_0(R) \to \mathrm{Pic}(R)$.

To fix this we need the **graded determinant**. Set

$$\mathcal{P}\mathrm{ic}^{\mathbb{Z}}(R) := \mathcal{P}\mathrm{ic}(R) \times \mathrm{Hom}_{\mathrm{cts}}(\mathrm{Spec}\, R; \mathbb{Z}),$$

then

$$\det{}_*(M) = (\det(M), \mathrm{rank}(M)),$$

does define a map of presheaves of $\mathbb{E}_\infty$-spaces:

$$\det{}_* : \iota\mathrm{Proj}^{\mathrm{fg}} \to \mathcal{P}\mathrm{ic}^{\mathbb{Z}};$$

since the target is grouplike it descends to a map:

$$\det{}_* : K_{\geqslant 0} \to \mathcal{P}\mathrm{ic}^{\mathbb{Z}},$$

whose effects on homotopy groups were explained in a previous lecture.

We want to define K-theory as a motivic spectrum some care has to be taken in doing this. Firstly, connective K-theory is not even a Zariski (let alone, Nisnevich) sheaf of spectra (on affine schemes) and neither is it $\mathbb{A}^1$-invariant.

Proposition 1.3.12. *Let* R *be a regular ring. The functor*

$$\mathrm{SmCAlg}_R \to \mathrm{Spt} \qquad S \mapsto K_{\geqslant 0}(S)$$

is an $\mathbb{A}^1$*-invariant Nisnevich sheaf.*

Indication of proof. Here are the key results one needs to prove Proposition 1.3.12. If S is a regular ring and $f \in S$ then we have a cofibre sequence of spectra

$$K_{\geqslant 0}(S \text{ on } f) \to K_{\geqslant 0}(S) \to K_{\geqslant 0}(S[\tfrac{1}{f}]).$$

The point here is that $K_{\geqslant 0}(S \text{ on } f)$ is the K-theory of a certain stable ∞-category (and not just some abstract fibre term) so it is important to extend the definition of K-theory to categories and not just rings. Furthermore if $S \to Q$ is étale and f is an element such that $Q \otimes_S S/f \xrightarrow{\simeq} Q/fQ$ then we have an equivalence on the fibre terms:

$$K_{\geqslant 0}(S \text{ on } f) \xrightarrow{\simeq} K_{\geqslant 0}(Q \text{ on } fQ);$$

this equivalence happens already on the categorical level. This implies Nisnevich descent for K-theory

There are many ways to prove that K-theory of a regular ring is $\mathbb{A}^1$-invariant; for example one can compare K-theory to G-theory or use Dévissage and a calculation of the K-theory of $\mathbb{P}^1$. □

To describe K-theory as a minimalist $\mathbb{A}^1$-invariant motivic cohomology theory, we use the following key element:

Construction 1.3.13 (The Bass element)**.** Consider the canonical map

$$\iota\mathrm{Proj}^{\mathrm{fg}}_{\mathbb{Z}[t,t^{-1}]} \to \mathrm{K}(\mathbb{Z}[t,t^{-1}]).$$

Then there is an element $\beta \in \pi_1(\iota\mathrm{Proj}^{\mathrm{fg}}_{\mathbb{Z}[t,t^{-1}]})$ which corresponds to the automorphism of the unit object, $\mathcal{O}_{\mathbb{G}_{m,\mathbb{Z}}}$ given by $\cdot t$. The image in K-theory

$$\beta_{\mathbb{Z}} \in \mathrm{K}_1(\mathbb{Z}[t,t^{-1}]),$$

will be called the **Bass element**, classified by a map $\mathrm{S}^1 \to \mathrm{K}(\mathbb{Z}[t,t^{-1}])$. Given any other ring R, we write β_{R} for the image of $\beta_{\mathbb{Z}}$ in $\mathrm{K}_1(\mathrm{R}[t,t^{-1}])$.

The Bass element clearly determines a map of presheaves of spectra on CAlg:

$$\beta : \mathbb{G}_m[1] \to \mathrm{K}_{\geqslant 0}.$$

By adjunction we have an induced map

$$\beta : \mathrm{K}_{\geqslant 0} \otimes \mathbb{G}_m[1] \to \mathrm{K}_{\geqslant 0}$$

and thus a map

$$\beta : \mathrm{K}_{\geqslant 0} \to \mathrm{map}(\mathbb{G}_m[1], \mathrm{K}_{\geqslant 0}).$$

Lemma 1.3.14. *On* SmCAlg*, we have an equivalence:*

$$\beta : \mathrm{K}_{\geqslant 0} \simeq \mathrm{map}(\mathbb{G}_m[1], \mathrm{K}_{\geqslant 0}).$$

Indication of proof. The key result is a calculation of K-theory of projective space (valid for all rings, and even base schemes)

$$\mathrm{K}_{\geqslant 0}(\mathbb{P}^1_{\mathrm{R}}) \simeq \mathrm{K}_{\geqslant 0}(\mathrm{R})\{\mathcal{O}\} \oplus \mathrm{K}_{\geqslant 0}(\mathrm{R})\{\mathcal{O}(-1) - \mathcal{O}\}. \tag{1.3.15}$$

We explain how to deduce the lemma from this result.

We contemplate the diagram

$$\begin{array}{ccccc} (\mathbb{A}^1_+)/\mathbb{A}^1 \smallsetminus 0_+ & \longleftarrow & \mathbb{P}^1_+/\mathbb{P}^1 \smallsetminus 0_+ & \longleftarrow & \mathbb{P}^1/\infty \\ \downarrow & & & \searrow & \downarrow{\scriptstyle \beta_{\mathcal{O}}} \\ \Sigma(\mathbb{G}_m/1) & & \xrightarrow{\quad\quad \beta \quad\quad} & & \mathrm{K}_{\geqslant 0}. \end{array}$$

Here, $\beta_{\mathcal{O}}$ classifies the virtual bundle $\mathcal{O}(-1)-\mathcal{O}$ and the map depends on a trivialization of $\mathcal{O}(-1)$ at ∞. We can extend such a trivialization over $\mathbb{P}^1 \smallsetminus 0$ which then produces the map emnating out of $\mathbb{P}^1_+/\mathbb{P}^1 \smallsetminus 0_+$. That the square commutes boils down to the fact that the trivialization of $\mathcal{O}(-1)$ over $\mathbb{P}^1 \smallsetminus 0$ and $\mathbb{P}^1 \smallsetminus \infty$ coincide over $\mathbb{G}_m$. Now, applying Propositions 1.1.5 and 1.3.12 we see that the map classifying the Bass element and the map classifying the second summand in (1.3.15) are homotopic after $\mathrm{L}_{\mathrm{Zar},\mathbb{A}^1}$. This proves the result.

□

Remark 1.3.16 (Connective K-theory of schemes)**.** In the above proof we have implicitly defined connective K-theory of schemes. This is done by right Kan extension:

$$\mathrm{K}_{\geqslant 0}(\mathrm{X}) := \lim_{\mathrm{Spec}\,\mathrm{R} \to \mathrm{X}} \mathrm{K}_{\geqslant 0}(\mathrm{R}),$$

where the limit is taken in the ∞-category of connective spectra/grouplike $\mathbb{E}_\infty$-spaces. If we fix a base regular ring R and took the right Kan extension from smooth commutative R-algebras to smooth R-schemes, then Proposition 1.1.6 proves that the resulting K-theory is also a Nisnevich sheaf.

Be warned that, in general (so if we do not restrict to smooth situations), if we had taken the right Kan extension in spectra, we might introduce the a version of K-theory with the wrong negative K-groups of schemes. The problem is that this latter version of K-theory does not satisfy the $\mathbb{P}^1$-bundle formula. To simultaneously ensure the correct values one has to proceed as in [TT90].

Example 1.3.17. If R is a regular ring, setting

$$\mathrm{KGL}(j) := \mathrm{K}[-j]$$

and using Lemmas 1.3.14 and 1.3.12 products a minimalist $\mathbb{A}^1$-invariant motivic cohomology theory. This is, however, not the good normalization. We will discuss this issues and more when we define KGL that also incorporates its multiplicative structure.

2. The "official" category of motivic spectra

Let S be a base scheme. In order to proceed further we will need to be able to elegantly manipulate motivic cohomology theories. The most important operation is probably taking tensor products. In algebraic topology, the search for a good "smash product" of spectra took the subject on various detours and turned out to be technically challenging before the dawn of higher category theory. Now that we have the latter, we can assemble the category of motivic cohomology theories into a nice symmetric monoidal ∞-category $\mathbf{SH}(\mathrm{S})$.

2.1. Universal properties, after Robalo-Voevodsky.

We begin with the symmetric monoidal ∞-category of pointed motivic spaces:

$$(\mathbf{H}(\mathrm{S})_\star := \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm_S})_\star, \wedge, \mathrm{S}_+).$$

In it, we have **Tate object** (aka pointed $\mathbb{P}^1$):

$$\mathbb{T} := \mathrm{cofib}(h(\mathrm{S}_+) \xrightarrow{\infty} h(\mathbb{P}^1_{\mathrm{S}})) \in \mathbf{H}(\mathrm{S})_\star$$

As already alluded to in the previous section, we have an equivalence in $\mathbf{H}(\mathrm{S})_\star$:

$$\mathrm{S}^1 \wedge \mathbb{G}_m \simeq \mathbb{T}.$$

General nonsense allows one to invert $\mathbb{T}$ and obtain a functor

$$\mathbf{H}(\mathrm{S})_\star \to \mathbf{SH}(\mathrm{S}) := \mathbf{H}(\mathrm{S})_\star[\mathbb{T}^{\wedge -1}].$$

We characterize the composite functor $\Sigma^\infty_{\mathbb{T}+} : \mathrm{Sm_S} \to \mathrm{Sm_{S+}} \to \mathbf{H}(\mathrm{S})_\star \to \mathbf{SH}(\mathrm{S})$ with a certain universal property:

Theorem 2.1.1 (Robalo-Voevodsky)**.** *The composite*

$$\Sigma^\infty_{\mathbb{T}+} : \mathrm{Sm}_{\mathrm{S}}^{\times} \to \mathbf{H}(\mathrm{S})_\star^{\wedge} \to \mathbf{SH}(\mathrm{S})^{\otimes}$$

satisfies the following universal property: if $\mathcal{C}^{\otimes}$ is a pointed, presentable symmetric monoidal ∞-category, then precomposition defines a fully faithful functor

$$\mathrm{Fun}^{\otimes,\mathrm{L}}(\mathbf{SH}(\mathrm{S}), \mathcal{C}) \hookrightarrow \mathrm{Fun}^{\otimes}(\mathrm{Sm_S}, \mathcal{C})$$

whose essential image contains those symmetric monoidal functors $\mathrm{F} : \mathrm{Sm_S} \to \mathcal{C}$ such that:

(1) F converts every Nisnevich square to a pullback square;
(2) F is $\mathbb{A}^1$-invariant;
(3) F takes $\mathbb{T}$ to an $\otimes$-invertible object.

Furthermore, any $\mathcal{C}$ admitting such a functor is necessarily stable.

Indication of proof. The key point is the notion of n-symmetricity. An object in a symmetric monoidal ∞-category $\mathrm{X} \in \mathcal{C}$ is said to be ***n*-symmetric** if the $\tau^n : \mathrm{X}\otimes\cdots\otimes\mathrm{X} \to \mathrm{X}\otimes\cdots\otimes\mathrm{X}$, the cyclic permutation on the n-tensor factors of X is homotopic to the identity. If X is n-symmetric for some n, then we can identify the formal inversion of X on $\mathcal{C}$ (which formally exists and is denoted by $\mathcal{C}[\mathrm{X}^{\otimes -1}]$) with a more concrete colimit of ∞-categories

$$\mathcal{C}[\mathrm{X}^{\otimes -1}] \simeq \operatorname{colim}(\mathcal{C} \xrightarrow{\mathrm{X}\otimes -} \mathcal{C} \xrightarrow{\mathrm{X}\otimes -} \cdots).$$

The latter formula is used to establish the basic properties of $\mathcal{C}[\mathrm{X}^{\otimes -1}]$ such a presentability and stability (in the case that $\mathcal{C} = \mathbf{H}(\mathrm{S})^{\wedge}_{\star}$). The object $\mathbb{T}$ is indeed 3-symmetric as one is encouraged to check by writing explicit homotopies.

See also the appendix to these notes for a "one categorical level down" discussion of what is going on. ☐

Sometimes it useful to relate Theorem 2.1.1 to the discussions from last time. We have the ∞-category of **$\mathbb{P}^1$-prespectra $\mathbf{PSp}_{\mathbb{T}}$**$(\mathrm{Sm}_{\mathrm{S}})$; roughly it is given by presheaves of spectra $\mathrm{E}(j)$ equipped with bonding maps $\mathrm{E}(j) \to \Omega_{\mathbb{P}^1}\mathrm{E}(j+1)$. While this ∞-category is relatively concrete, it does not admit a suitable symmetric monoidal structure. To convert a $\mathbb{P}^1$-prespectrum to a $\mathbb{P}^1$-spectrum we set

$$(\mathrm{L}_{\mathbb{P}^1}\mathrm{E}(\star))(j) = \operatorname*{colim}_n \Omega^n_{\mathbb{P}^1}\mathrm{E}(n+j)$$

where the transition maps are induced by the bonding maps.

Corollary 2.1.2. *The endofunctor is a localization*

$$\mathrm{L}_{\mathbb{P}^1,\mathrm{Nis},\mathbb{A}^1} : \mathbf{PSp}_{\mathbb{T}}(\mathrm{Sm}_{\mathrm{S}}) \to \mathbf{PSp}_{\mathbb{T}}(\mathrm{Sm}_{\mathrm{S}})$$

has essential image $\mathbf{SH}(\mathrm{S})$*.*

Definition 2.1.3 (The S-motive of Y)**.** If $\mathrm{Y} \in \mathrm{Sm}_{\mathrm{S}}$, we set

$$\mathrm{M}_{\mathrm{S}}(\mathrm{Y}) := \Sigma^{\infty}_{\mathbb{T}+}\mathrm{Y}.$$

If the context we clear, we often suppress the "S." We call $\mathrm{M}_{\mathrm{S}}(\mathrm{Y})$ the **S-motive of** Y.

2.2. **Calculating E-cohomology.** To calculate the E-cohomology of an smooth S-scheme X, we adopt the following notation. We set

$$\mathbb{S}^{i,j} := \Sigma^{i-2j}\mathbb{T}^{\otimes j} \simeq \Sigma^{i-j}\mathbb{G}_m^{\otimes j};$$

and for any $\mathrm{E} \in \mathbf{SH}(\mathrm{S})$ write

$$\Sigma^{i,j}\mathrm{E} := \mathbb{S}^{i,j} \otimes \mathrm{E}.$$

If $\mathrm{E} \in \mathbf{SH}(\mathrm{S})$, then the **E-cohomology** of Y is set to be

$$\mathrm{E}^{i,j}(\mathrm{X}) := \pi_0 \operatorname{Maps}_{\mathbf{SH}(\mathrm{S})}(\mathrm{M}_{\mathrm{S}}(\mathrm{Y}), \Sigma^{i,j}\mathrm{E}).$$

Under this convention, if we write $\mathrm{E} = \{\mathrm{E}(j)\}$ where $\mathrm{E}(j)$ is an $\mathbb{A}^1$-invariant motivic cohomology theory we have that

$$\mathrm{E}^{i,j}(\mathrm{X}) \simeq \pi_0(\mathrm{E}(j)[i-2j](\mathrm{X})).$$

For example, we have

$$(\mathrm{H}_{\text{ét}}\mu_{p^r})^{i,j}(\mathrm{Y}) = \mathrm{H}^i_{\text{ét}}(\mathrm{Y}; \mu_{p^r}^{\otimes j})$$

and

$$\mathrm{KGL}^{i,j}(\mathrm{X}) = \mathrm{K}_{2j-i}(\mathrm{X}).$$

3. The slice filtration

Having constructed a nice symmetric monoidal stable ∞-category category $\mathbf{SH}(\mathrm{S})$, we are now equipped to discuss the slice filtration.

Definition 3.0.1. We define subcategories

$$\cdots \mathbf{SH}(\mathrm{S})^{\mathrm{eff}}(n) \subset \mathbf{SH}(\mathrm{S})^{\mathrm{eff}}(n-1) \subset \cdots \subset \mathbf{SH}(\mathrm{S})^{\mathrm{eff}}(0) = \mathbf{SH}(\mathrm{S})^{\mathrm{eff}} \subset \mathbf{SH}(\mathrm{S})^{\mathrm{eff}}(-1) \subset \cdots \mathbf{SH}(\mathrm{S})$$

as follows: $\mathbf{SH}(\mathrm{S})^{\mathrm{eff}}(n)$ is the full subcategory $\mathbf{SH}(\mathrm{S})$ generated by $\{\mathrm{M_S(Y)}[-m] \otimes \mathbb{T}^{\otimes n} : m \in \mathbb{Z}, \mathrm{Y} \in \mathrm{Sm_S}\}$ under colimits and extensions. An object $\mathrm{E} \in \mathbf{SH}(\mathrm{S})^{\mathrm{eff}}(n)$ is called a **n-effective spectrum** and if $n = 0$ we call E a **effective spectrum**.

We note that $\mathbf{SH}(\mathrm{S})^{\mathrm{eff}}(n)$ defines the non-negative part of a t-structure with right orthogonal the subcategory of those spectra F such that

$$\mathrm{Maps}(\mathrm{E}, \mathrm{F}) \simeq * \qquad \forall \mathrm{E} \in \mathbf{SH}(\mathrm{S})^{\mathrm{eff}}(n).$$

In this case, we say that F is **n-coeffective**. However, since $\mathbf{SH}(\mathrm{S})^{\mathrm{eff}}(n)$ is defined to contain all shifts, the t-structure is not very interesting as it has zero heart. The following lemma records basic properties of these categories. Its formal properties are:

Lemma 3.0.2. *Let* X *be a scheme,* $n \in \mathbb{Z}$*.*

(1) $\mathbf{SH}(\mathrm{S})^{\mathrm{eff}}(n) \subset \mathbf{SH}(\mathrm{S})$ *is a presentable, stable subcategory which is closed under* $\otimes$*.*
(2) the inclusion $\iota^n : \mathbf{SH}(\mathrm{S})^{\mathrm{eff}}(n) \subset \mathbf{SH}(\mathrm{S})$ *admits a right adjoint*

$$r^n : \mathbf{SH}(\mathrm{S}) \to \mathbf{SH}(\mathrm{S})^{\mathrm{eff}}(n).$$

(3) Set

$$f^n_{\mathrm{slice}}\mathrm{E} := \iota^n r^n \mathrm{E}.$$

The construction

$$\mathrm{E} \in \mathbf{SH}(\mathrm{S}) \mapsto \cdots \to f^n_{\mathrm{slice}}\mathrm{E} \to f^{n-1}_{\mathrm{slice}}\mathrm{E} \cdots f^0_{\mathrm{slice}}\mathrm{E} \to f^{-1}_{\mathrm{slice}}\mathrm{E} \cdots =: f^{\star}_{\mathrm{slice}}\mathrm{E}$$

promotes to a lax symmetric monoidal functor

$$\mathbf{SH}(\mathrm{S}) \to \mathbf{SH}(\mathrm{S})^{(\mathbb{Z}, \geqslant)^{\mathrm{op}}}.$$

Appendix A. Attaching cells

Let R be a commutative ring and $f \in \mathrm{R}$ be a nonzero divisor. Then how does one produce the localization $\mathrm{R}[\frac{1}{f}]$? On the one hand it admits a universal property: it is the initial commutative ring under R such that under any ring map $\mathrm{R} \to \mathrm{S}$, f is sent to an invertible element. The existence of such a ring can be proved by abstract nonsense. Nonetheless, it is helpful to have a more explicit model. Consider the colimit, taken in R-modules

$$\mathrm{R} \xrightarrow{\cdot f} \mathrm{R} \xrightarrow{\cdot f} \mathrm{R} \xrightarrow{\cdot f} \cdots ...$$

Then the colimit of this diagram is an R-module and there is an candidate inverse of f. Indeed, scalar multiplication by f on R is induced by the map between diagrams

$$\begin{array}{ccccc} \mathrm{R} & \xrightarrow{\cdot f} & \mathrm{R} & \xrightarrow{\cdot f} & \cdots \\ \downarrow{\scriptstyle \cdot f} & & \downarrow{\scriptstyle \cdot f} & & \\ \mathrm{R} & \xrightarrow{\cdot f} & \mathrm{R} & \xrightarrow{\cdot f} & \cdots \end{array} \qquad \begin{array}{cc} \mathrm{R} \xrightarrow{\cdot f} \cdots \\ \downarrow{\scriptstyle \cdot f} \\ \mathrm{R} \xrightarrow{\cdot f} \cdots , \end{array}$$

and a candidate inverse is given by the following map between diagrams; a process that "staggers" the above diagram:

$$\begin{array}{ccccccc} R & \xrightarrow{\cdot f} & R & \xrightarrow{\cdot f} & \cdots & \qquad & R \xrightarrow{\cdot f} \cdots \\ & \overset{\text{id}}{\searrow} & \downarrow{\scriptstyle \text{id}} & \overset{\text{id}}{\searrow} & & & \downarrow{\scriptstyle \text{id}} \\ R & \xrightarrow{\cdot f} & R & \xrightarrow{\cdot f} & \cdots & \qquad & R \xrightarrow{\cdot f} \cdots , \end{array}$$

Indeed, stacking the two diagrams both ways gives:

$$\begin{array}{cccccccc} R & \xrightarrow{\cdot f} & R & \xrightarrow{\cdot f} & R\cdots & & \qquad & R \xrightarrow{\cdot f} \cdots \\ & \overset{f}{\searrow} & \downarrow{\scriptstyle \text{id}} & \overset{f}{\searrow} & \downarrow{\scriptstyle \text{id}} & \overset{f}{\searrow} & & \\ R & \xrightarrow{\cdot f} & R & \xrightarrow{\cdot f} & R & & \qquad & R \xrightarrow{\cdot f} \cdots , \end{array}$$

so that the induced map on the level of colimits gives the identity map.

In higher algebra, the commutativity of diagrams is data and not property. In particular if R is a $\mathbb{E}_\infty$ (in whatever context), it only makes sense to ask for (invertible) 2-cells:

$$\begin{array}{ccc} R & \xrightarrow{\cdot f} & R \\ {\scriptstyle \cdot g}\downarrow & \overset{\tau_{f,g}}{\swarrow\!\!\swarrow} & \downarrow{\scriptstyle \cdot g} \\ R & \xrightarrow[\cdot f]{} & R. \end{array}$$

This poses some challenge in inverting elements in higher algebra.

Suppose that want to invert f in higher algebra. We begin by contemplating the higher analog of multiplication by f:

$$\begin{array}{ccccccc} R & \xrightarrow{\cdot f} & R & \xrightarrow{\cdot f} & R & \qquad & R \xrightarrow{\cdot f} \cdots \\ {\scriptstyle \cdot f}\downarrow & \overset{\tau}{\swarrow\!\!\swarrow} & {\scriptstyle \cdot f}\downarrow & \overset{\tau}{\swarrow\!\!\swarrow} & \downarrow & & {\scriptstyle \cdot f}\downarrow \\ R & \xrightarrow{\cdot f} & R & \xrightarrow{\cdot f} & R & \qquad & R \xrightarrow{\cdot f} \cdots , \end{array}$$

.

Its candidate inverse is easily constructed because we are just sticking in the identity cells.

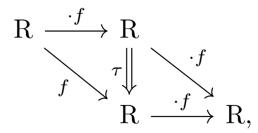
$$\begin{array}{ccccccc} R & \xrightarrow{\cdot f} & R & \xrightarrow{\cdot f} & \cdots & \qquad & R \xrightarrow{\cdot f} \cdots \\ & \overset{\text{id}}{\searrow} & \downarrow{\scriptstyle \text{id}} & \overset{\text{id}}{\searrow} & & & \downarrow{\scriptstyle \text{id}} \\ R & \xrightarrow{\cdot f} & R & \xrightarrow{\cdot f} & \cdots & \qquad & R \xrightarrow{\cdot f} \cdots , \end{array}$$

However, when we start stacking we get two cells in between the parallelograms:

$$\begin{array}{ccccccccc} R & \xrightarrow{\cdot f} & R & \xrightarrow{\cdot f} & R & \xrightarrow{\cdot f} & R & \qquad & R \xrightarrow{\cdot f} \cdots \\ & \overset{f}{\searrow} & {\scriptstyle \tau}\Downarrow & \overset{f}{\searrow} & {\scriptstyle \tau}\Downarrow & \overset{\cdot f}{\searrow} & & & {\scriptstyle \tau}\Downarrow \\ R & \xrightarrow{\cdot f} & R & \xrightarrow{\cdot f} & R & \xrightarrow{\cdot f} & R & \qquad & R \xrightarrow{\cdot f} \cdots . \end{array} \tag{A.0.1}$$

To create an inverse, we need the resulting colimit to be homotopic to the identity. To assure this in the strongest way possible, we can ask that τ is the identity cell. But there are a couple of other ways to ensure this:

(1) Concentrating on the diagram



we see that τ is classified by a loop based on f^2, hence τ defines a map $\mathrm{S}^1 \to \mathrm{R}$ at $f^2 \in \pi_0(\mathrm{R})$. By the $\mathbb{E}_\infty$-structure of R, we know that $\tau^2 \simeq \mathrm{id}$ and thus this gives a map $\Sigma_2 \to \pi_1(\mathrm{R}, f^2)$. We can ask that this map sends τ, i.e., the non-identity element to $1 \in \pi_1(\mathrm{R}, f^2)$.

(2) In fact, we can just ask that the map
$$\Sigma_2 \to \pi_1(\mathrm{R}, f^2) \to \pi_1(\mathrm{T}_f\mathrm{R}, f^2)$$
is trivial.

(3) Actually we ask for even less: the horizontal compositum of (A.0.1) and the $\mathbb{E}_\infty$-structure on R gives rise to a map
$$\Sigma_n \to \pi_1(\mathrm{R}, f^n) \to \pi_1(\mathrm{T}_f\mathrm{R}, f^n)$$
We need only ask that the cyclic permutation $(12 \cdots n)$ gets sent to the identity.

Pushing this argument one learns that if R is a $\mathbb{E}_\infty$-monoid, then the map $\mathrm{T}_\infty\mathrm{R} \to \mathrm{R}^{\mathrm{grp}}$ is an equivalence if and only if for each $f \in \pi_0(\mathrm{R})$ there exists some n (which may vary as n varies) such that the map
$$\Sigma_n \to \pi_n(\mathrm{T}_\infty\mathrm{R}, f^n)$$
has the cyclic permutation $(12 \cdots n)$ in the kernel. This is the moment when there is cell attaching. In general, filtered colimits commute with homotopy groups, so the homotopy of $\mathrm{T}_\infty\mathrm{R}$ will just be a colimit of the homotopy of R's: in particular if $\pi_k(\mathrm{R}) = 0$, then $\pi_k(\mathrm{T}_\infty\mathrm{R})$ is still zero. But in order to guarantee the conditions above, we need to attach cells which can affect add new homotopy groups to $\mathrm{T}_\infty\mathrm{R}$.

In fact, the correct hypothesis is that we want π_1 to be **hypoabelian**: it has no nontrivial perfect subgroups. Recall that a group is perfect if it is equals to its own commutator subgroup: $\mathrm{G} = [\mathrm{G}, \mathrm{G}]$. Note that the process of passing froM $\mathrm{T}_\infty\mathrm{R}$ to $\mathrm{R}^{\mathrm{grp}}$ is not quite as simple as attaching one cell for each generator: we have to do this functorially and we have to ensure that it is the initial thing whose π_1 is hypoabelian.

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