Zero cycles on singular varieties

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(joint work with Matthew Morrow)

The goal of this talk is to explain a short cohomological approach of most cases of the following result. More details will appear in an upcoming paper, joint with Matthew Morrow.

Theorem 1 (Levine, Krishna–Srinivas, Krishna)**.** *Let k be an algebraically closed field and A a reduced, finite type k-scheme of dimension $d \geq 2$. Then the subgroup $F^d\mathrm{K}_0(\mathrm{A}) \subset \mathrm{K}_0(\mathrm{A})$ is torsionfree.*

Here, $F^d\mathrm{K}_0(\mathrm{A})$ is the subgroup of $\mathrm{K}_0(\mathrm{A})$ generated by classes of smooth points of codimension d points of Spec(A). Classically, it is considered as replacement for the Chow group of zero cycles for schemes which are not necessarily smooth over a field.

Remark 2. *For some context, Theorem 1 is an answer to a question posed by Murthy in* [9, Open Question (2.12)]*, in the context of splitting problems for vector bundles over affine varieties. In particular, Theorem 1 shows that the top chern class in $F^d\mathrm{K}_0(\mathrm{A})$ is a complete obstruction to splitting off a trivial rank one summand in a rank d vector bundle over* Spec(A)*. For torsion prime to the characteristic, Theorem 1 was proved by Levine in an unpublished manuscript* [8]*. A major breakthrough was made for normal varieties in* [7] *and a proof of the full conjecture can be found in* [6]*. Our approach here is different and uses vanishing results for étale/syntomic cohomology of affine varieties which are algebro-geometric analogs of the Andreotti-Frankel theorem in topology.*

Our approach to this result relies on the extension of motivic cohomology to all equicharacteristic quasicompact, quasiseparated schemes introduced in [3]. Let us review parts of this theory that we will use in the proof.

Theorem 3. *Let k be a field. There is a functorial, multiplicative, $\mathbb{N}$-indexed and complete filtration on the K-theory of an algebraic variety X over k:*

$$\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K(X)} \to \mathrm{K(X)}.$$

Writing its graded pieces and associated cohomology as:

$$\mathbb{Z}(j)^{\mathrm{mot}}(X) := \mathrm{gr}^j_{\mathrm{mot}}\mathrm{K(X)[-2j]} \qquad \mathrm{H}^{\mathrm{i}}_{\mathrm{mot}}(\mathrm{X}, \mathbb{Z}(\mathrm{j})) := \mathrm{H}^{\mathrm{i}}(\mathbb{Z}(\mathrm{j})^{\mathrm{mot}}(\mathrm{X})),$$

there is a functorial motivic-to-syntomic *comparison map*

$$\mathbb{Z}(j)^{\mathrm{mot}}(X)/p \to \mathbb{F}_p(j)^{\mathrm{syn}}(X)^1$$

satisfying the following properties

[1] Let us recall that if p is invertible in k then syntomic cohomology of X is its étale cohomology, with an appropriate Tate twist. On the other hand, if $p = 0$ in k then weight-j mod-p syntomic cohomology is described as the left Kan extension of the functor $X \mapsto R\Gamma_{\text{ét}}(X; \Omega^j_{\log})[-j]$.

(1) *if p is invertible in k, then we have a functorial equivalence under the motivic-to-syntomic comparison map:*

$$\mathbb{Z}(j)^{\mathrm{mot}}(X)/p \simeq L_{\mathrm{cdh}}\tau^{\leq j} R\Gamma_{\acute{e}t}(-, \mu_p^{\otimes j})(X).$$

(2) *If $p = 0$ in k then we have a cartesian square*

$$\begin{array}{ccc} \mathbb{Z}(j)^{\mathrm{mot}}(X)/p & \longrightarrow & \mathbb{F}_p(j)^{\mathrm{syn}}(X) \\ \downarrow & & \downarrow \\ R\Gamma_{\mathrm{cdh}}(X, \Omega^j_{\log})[-j] & \longrightarrow & R\Gamma_{\acute{e}t}(X, \Omega^j_{\log})[-j]. \end{array}$$

(3) *Weibel vanishing: for $j \geq 0$ we have*

$$H^{>j+\dim(X)}_{\mathrm{mot}}(X, \mathbb{Z}(j)) = 0.$$

(4) *Nisnevich-locally, the presheaf $H^j_{\mathrm{mot}}(-, \mathbb{Z}(j))$ identifies with Gabber-Kerz's improved Milnor K-theory of a local ring.*

(5) *After rationalisation, the filtration functorially splits. In particular, the resulting spectral sequence rationally degenerates.*

Key to a cohomological approach to Theorem 1 is the following result, which can be regraded as a cohomological version of the vanishing theorems of Suslin [10] and Geisser [4].

Theorem 4. *Let X be an scheme of finite type over k, an algebraically closed field. Then for $j \geq \dim(X)$ the motivic-to-syntomic comparison map*

$$\mathbb{Z}(j)^{\mathrm{mot}}(X)/p \to \mathbb{F}_p(j)^{\mathrm{syn}}(X)$$

is an equivalence.

Proof. First, let us assume that p is invertible in k. Then, by Theorem 3(1), the cofiber of the motivic-to-syntomic comparison map in weight j is equivalent to the presheaf $L_{\mathrm{cdh}}\tau^{>j} R\Gamma_{\text{ét}}(-, \mathbb{F}_p)$. To check that this latter presheaf is zero in the stated range, it suffices to prove that for proper cdh cover $Y \to X$ where $\mathrm{dim(Y)} \leq \mathrm{dim(X)}$, the presheaf $\tau^{>j} R\Gamma_{\text{ét}}(-, \mathbb{F}_p)$ is Nisnevich-locally trivial. However, for any affine scheme Spec(A), étale over Y we have that $R\Gamma_{\text{ét}}(\mathrm{Spec(A)}, \mathrm{\mathbb{F}_p})$ is concentrated in degrees $\leq \mathrm{dim(A)} \leq \mathrm{dim(X)}$ by Artin's vanishing theorem [1, Corollaire XIV.3.2].

Next, assume that $p = 0$ in k. This time, by Theorem 3(2), the cofiber of the motivic-to-syntomic comparison map in weight j is equvialent to the presheaf $R\Gamma_{\mathrm{cdh}}(-, \widetilde{\nu}(j))$ where $\widetilde{\nu}(j)$ is the j-th Artin–Schreier obstruction [3, Definition 4.29]. In the notation of the previous paragraph, it then suffices to prove that $\widetilde{\nu}(j)$ is Nisnevich-locally trivial on Y for $j \geq \dim(X)$. Let $y \in Y$, we claim that $\widetilde{\nu}(j)(\mathcal{O}^h_{Y,y}) = 0$. We now appeal to the rigidity theorem of Antieau–Mathew–Morrow–Nikolaus [2, Theorem 5.2] which shows that $\widetilde{\nu}(j)(\mathcal{O}^h_{Y,y}) \cong \widetilde{\nu}(j)(\kappa(y))$ where $\kappa(y)$ is the residue field of $y \in Y$. Now, $\kappa(y)$ is transcendence degree $\leq \mathrm{dim(Y)} \leq \mathrm{dim(X)}$ over k; that the the Artin–Schreier obstruction vanishes is a consequence of the definition of Kato's p-dimension and the inequality of Kato–Kuzumaki [5, Corollary 2].

$\square$

Remark 5. *Let X be a quasiprojective variety over k and $z^j(X,\bullet)$ be Bloch's cycle complex so that its* higher Chow groups *(with coefficients) are defined as the homology groups*

$$\mathrm{CH}^j(X,i;\mathbb{Z}/m) =: H_i(z^j(X,\bullet)\otimes^{\mathbb{L}}\mathbb{Z}/m).$$

Then Suslin (in characteristic zero [10], $j \geq \dim(X)$) and Geisser (in arbitrary characteristic [4] for $j = \dim(X)$) proved that there is an isomorphism

$$\mathrm{CH}^j(X,i;\mathbb{Z}/m) \cong H^{2(\dim(X)-i)+j}_{\mathrm{syn}}(X;\mathbb{Z}/m(d-j))^{\sharp},$$

where X is an equidimensional quasiprojective variety over an algebraically closed field and $\sharp$ denote Pontrjagin dual (take maps into $\mathbb{Q}/\mathbb{Z}$).

It is expected that Bloch's cycle complex is a model for Borel–Moore motivic homology so that, in particular, it admits a functorial action of motivic cohomology which refines the action of K-theory on G-theory. Under this expected relationship, the above isomorphism is a Borel–Moore counterpart to Theorem 4.

Equipped with Theorem 4, we proceed to reduce Theorem 1 to a problem in syntomic cohomology. First, we only assume that X is a k-variety (not necessarily affine). Thanks to Theorem 3(3), the motivic spectral sequence for X gives rise to an edge homomorphism

$$\mathrm{edge}: H^{2d}_{\mathrm{mot}}(X,\mathbb{Z}(d)) \to \mathrm{K}_0(X).$$

A key property of this edge homomorphism is that it covers $F^d\mathrm{K}_0(X)$ in the sense that $F^d\mathrm{K}_0(X) \subset \mathrm{Image(edge)}$. The proof of this involves the construction of a *cycle class map*; this is a functorial map $Z_0(X_{\mathrm{sm}}) \to H^{2d}_{\mathrm{mot}}(X,\mathbb{Z}(d))$ from the abelian group of zero cycles on the smooth locus of a k-variety X. It has the property that after post-composing with the edge map, it sends the class of a point $[x] \in Z_0(X_{\mathrm{sm}})$ to the same-named element in $\mathrm{K}_0(\mathrm{X})$.

Combined with Theorem 3(5) which implies that the edge map is a rational injection, Theorem 1 will follow once we know that $H^{2d}_{\mathrm{mot}}(X,\mathbb{Z}(d))$ is torsionfree. This we may check one prime at a time. Let p be a prime then plugging in Theorem 4, the Bockstein sequence looks like

$$H^{2d-1}_{\mathrm{mot}}(X,\mathbb{Z}(d)) \to H^{2d-1}_{\mathrm{syn}}(X;\mathbb{F}_p(d)) \xrightarrow{\delta} H^{2d}_{\mathrm{mot}}(X,\mathbb{Z}(d)) \xrightarrow{\cdot p} H^{2d}_{\mathrm{mot}}(X,\mathbb{Z}(d)) \to \cdots$$

Now we use that X is affine. If p is invertible, then $H^{2d-1}_{\mathrm{syn}}(X;\mathbb{F}_p(d)) = H^{2d-1}_{\text{ét}}(X;\mathbb{F}_p) = 0$ as soon as $2d-1 \geq d+1$ by Artin's vanishing theorem. In other words, $\cdot p$ is injective when $d \geq 2$. If k is characteristic p, then we have that $H^{2d-1}_{\mathrm{syn}}(X;\mathbb{F}_p(d)) = 0$ whenever $2d-1 \geq d+2$ by [2, Theorem G]. In other words, $\cdot p$ is injective whenever $d \geq 3$.

When $p = 0$ in k and $d = 2$ the result is equivalent to the surjectivity of the map $H^3_{\mathrm{syn}}(X,\mathbb{Z}_p(2)) \to H^3_{\mathrm{syn}}(X,\mathbb{F}_p(2))$ induced by mod-p reduction. We are working on a direct proof of this surjectivity.

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